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# Isabelle/UTP: Mechanised Theory Engineering for Unifying Theories of Programming

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## Abstract

Isabelle/UTP is a mechanised theory engineering toolkit based on Hoare and He’s Unifying Theories of Programming (UTP). UTP enables the creation of denotational, algebraic, and operational semantics for different programming languages using an alphabetised relational calculus. We provide a semantic embedding of the alphabetised relational calculus in Isabelle/HOL, including new type definitions, relational constructors, automated proof tactics, and accompanying algebraic laws. Isabelle/UTP can be used to both capture laws of programming for different languages, and put these fundamental theorems to work in the creation of associated verification tools, using calculi like Hoare logics. This document describes the relational core of the UTP in Isabelle/HOL.

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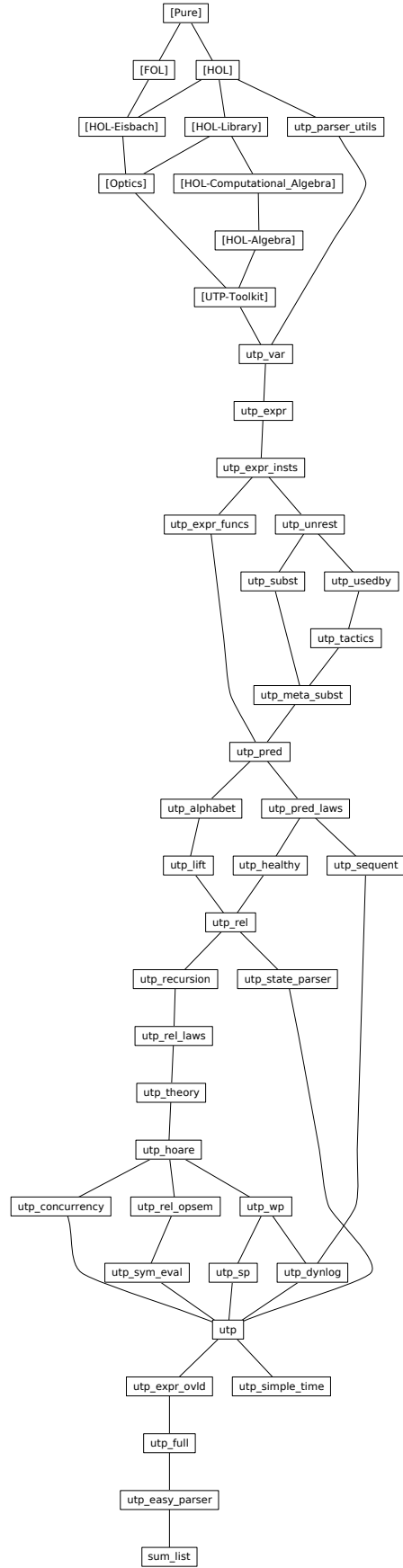
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# 1 Introduction

This document contains the description of our mechanisation of Hoare and He’s *Unifying Theories of Programming* [22, 7] (UTP) in Isabelle/HOL. UTP uses the “programs-as-predicates” approach, pioneered by Hehner [20, 18, 19], to encode denotational semantics and facilitate reasoning about programs. It uses the alphabetised relational calculus, which combines predicate calculus and relation algebra, to denote programs as relations between initial variables ( $x$ ) and their subsequent values ( $x'$ ). Isabelle/UTP<sup>1</sup> [16, 28, 15] semantically embeds this relational calculus into Isabelle/HOL, which enables application of the latter’s proof facilities to program verification. For an introduction to UTP, we recommend two tutorials [6, 7], and also the UTP book [22].

The Isabelle/UTP core mechanises most of definitions and theorems from chapters 1, 2, 4, and 7 of [22], and some material contained in chapters 5 and 10. This essentially amounts to alphabetised predicate calculus, its core laws, the UTP theory infrastructure, and also parallel-by-merge [22, chapter 5], which adds concurrency primitives. The Isabelle/UTP core does not contain the theory of designs [6] and CSP [7], which are both represented in their own theory developments.

A large part of the mechanisation, however, is foundations that enable these core UTP theories. In particular, Isabelle/UTP builds on our implementation of lenses [16, 14], which gives a formal semantics to state spaces and variables. This, in turn, builds on a previous version of Isabelle/UTP [9, 10], which provided a shallow embedding of UTP by using Isabelle record types to represent alphabets. We follow this approach and, additionally, use the lens laws [11, 16] to characterise well-behaved variables. We also add meta-logical infrastructure for dealing with free variables and substitution. All this, we believe, adds an additional layer rigour to the UTP.

The alphabets-as-types approach does impose a number of theoretical limitations. For example, alphabets can only be extended when an injection into a larger state-space type can be exhibited. It is therefore not possible to arbitrarily augment an alphabet with additional variables, but new types must be created to do this. This is largely because as in previous work [9, 10], we actually encode state spaces rather than alphabets, the latter being implicit. Namely, a relation is typed by the state space type that it manipulates, and the alphabet is represented by collection of lenses into this state space. This aspect of our mechanisation is actually much closer to the relational program model in Back’s refinement calculus [3].

The pay-off is that the Isabelle/HOL type checker can be directly applied to relational constructions, which makes proof much more automated and efficient. Moreover, our use of lenses mitigates the limitations by providing meta-logical style operators, such as equality on variables, and alphabet membership [16]. Isabelle/UTP can therefore directly harness proof automation from Isabelle/HOL, which allows its use in building efficient verification tools [13, 12]. For a detailed discussion of semantic embedding approaches, please see [28].

In addition to formalising variables, we also make a number of generalisations to UTP laws. Notably, our lens-based representation of state leads us to adopt Back’s approach to both assignment and local variables [3]. Assignment becomes a point-free operator that acts on state-space update functions, which provides a rich set of algebraic theorems. Local variables are represented using stacks, unlike in the UTP book where they utilise alphabet extension.

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<sup>1</sup>Isabelle/UTP website: <https://www.cs.york.ac.uk/circus/isabelle-utp/>



We give a summary of the main contributions within the Isabelle/UTP core, which can all be seen in the table of contents.

1. Formalisation of variables and state-spaces using lenses [16];
2. an expression model, together with lifted operators from HOL;
3. the meta-logical operators of unrestriction, used-by, substitution, alphabet extrusion, and alphabet restriction;
4. the alphabetised predicate calculus and associated algebraic laws;
5. the alphabetised relational calculus and associated algebraic laws;
6. proof tactics for the above based on interpretation [23];
7. a formalisation of UTP theories using locales [4] and building on HOL-Algebra [5];
8. Hoare logic [21] and dynamic logic [17];
9. weakest precondition and strongest postcondition calculi [8];
10. concurrent programming with parallel-by-merge;
11. relational operational semantics.

## 2 UTP Variables

```
theory utp-var
imports
  UTP-Toolkit.utp-toolkit
  utp-parser-utils
begin
```

In this first UTP theory we set up variables, which are built on lenses [11, 16]. A large part of this theory is setting up the parser for UTP variable syntax.

### 2.1 Initial syntax setup

We will overload the square order relation with refinement and also the lattice operators so we will turn off these notations.

```
purge-notation
  Order.le (infixl  $\sqsubseteq_1$  50) and
  Lattice.sup ( $\sqcup_1$ - [90] 90) and
  Lattice.inf ( $\sqcap_1$ - [90] 90) and
  Lattice.join (infixl  $\sqcup_1$  65) and
  Lattice.meet (infixl  $\sqcap_1$  70) and
  Set.member (op :) and
  Set.member ((-/ : -) [51, 51] 50) and
  disj (infixr | 30) and
  conj (infixr & 35)
```

```
declare fst-vwb-lens [simp]
declare snd-vwb-lens [simp]
declare comp-vwb-lens [simp]
```

```

declare lens-indep-left-ext [simp]
declare lens-indep-right-ext [simp]
declare lens-comp-quotient [simp]
declare plus-lens-distr [THEN sym, simp]

```

## 2.2 Variable foundations

This theory describes the foundational structure of UTP variables, upon which the rest of our model rests. We start by defining alphabets, which following [9, 10] in this shallow model are simply represented as types  $'\alpha$ , though by convention usually a record type where each field corresponds to a variable. UTP variables in this frame are simply modelled as lenses  $'a \Longrightarrow '\alpha$ , where the view type  $'a$  is the variable type, and the source type  $'\alpha$  is the alphabet or state-space type.

We define some lifting functions for variables to create input and output variables. These simply lift the alphabet to a tuple type since relations will ultimately be defined by a tuple alphabet.

**definition** *in-var* ::  $('a \Longrightarrow '\alpha) \Rightarrow ('a \Longrightarrow '\alpha \times '\beta)$  **where**  
*[lens-defs]: in-var x = x ;<sub>L</sub> fst<sub>L</sub>*

**definition** *out-var* ::  $('a \Longrightarrow '\beta) \Rightarrow ('a \Longrightarrow '\alpha \times '\beta)$  **where**  
*[lens-defs]: out-var x = x ;<sub>L</sub> snd<sub>L</sub>*

Variables can also be used to effectively define sets of variables. Here we define the the universal alphabet ( $\Sigma$ ) to be the bijective lens  $1_L$ . This characterises the whole of the source type, and thus is effectively the set of all alphabet variables.

**abbreviation** (*input*) *univ-alpha* ::  $('a \Longrightarrow '\alpha) (\Sigma)$  **where**  
*univ-alpha*  $\equiv 1_L$

The next construct is vacuous and simply exists to help the parser distinguish predicate variables from input and output variables.

**definition** *pr-var* ::  $('a \Longrightarrow '\beta) \Rightarrow ('a \Longrightarrow '\beta)$  **where**  
*[lens-defs]: pr-var x = x*

## 2.3 Variable lens properties

We can now easily show that our UTP variable construction are various classes of well-behaved lens .

**lemma** *in-var-weak-lens* [*simp*]:  
 $weak\text{-}lens\ x \Longrightarrow weak\text{-}lens\ (in\text{-}var\ x)$   
**by** (*simp add: comp-weak-lens in-var-def*)

**lemma** *in-var-semi-uvar* [*simp*]:  
 $mwb\text{-}lens\ x \Longrightarrow mwb\text{-}lens\ (in\text{-}var\ x)$   
**by** (*simp add: comp-mwb-lens in-var-def*)

**lemma** *pr-var-weak-lens* [*simp*]:  
 $weak\text{-}lens\ x \Longrightarrow weak\text{-}lens\ (pr\text{-}var\ x)$   
**by** (*simp add: pr-var-def*)

**lemma** *pr-var-mwb-lens* [*simp*]:  
 $mwb\text{-}lens\ x \Longrightarrow mwb\text{-}lens\ (pr\text{-}var\ x)$   
**by** (*simp add: pr-var-def*)

**lemma** *pr-var-vwb-lens* [simp]:  
 $vwb\text{-}lens\ x \implies vwb\text{-}lens\ (pr\text{-}var\ x)$   
**by** (simp add: pr-var-def)

**lemma** *in-var-uvar* [simp]:  
 $vwb\text{-}lens\ x \implies vwb\text{-}lens\ (in\text{-}var\ x)$   
**by** (simp add: in-var-def)

**lemma** *out-var-weak-lens* [simp]:  
 $weak\text{-}lens\ x \implies weak\text{-}lens\ (out\text{-}var\ x)$   
**by** (simp add: comp-weak-lens out-var-def)

**lemma** *out-var-semi-uvar* [simp]:  
 $mwb\text{-}lens\ x \implies mwb\text{-}lens\ (out\text{-}var\ x)$   
**by** (simp add: comp-mwb-lens out-var-def)

**lemma** *out-var-uvar* [simp]:  
 $vwb\text{-}lens\ x \implies vwb\text{-}lens\ (out\text{-}var\ x)$   
**by** (simp add: out-var-def)

Moreover, we can show that input and output variables are independent, since they refer to different sections of the alphabet.

**lemma** *in-out-indep* [simp]:  
 $in\text{-}var\ x \bowtie out\text{-}var\ y$   
**by** (simp add: lens-indep-def in-var-def out-var-def fst-lens-def snd-lens-def lens-comp-def)

**lemma** *out-in-indep* [simp]:  
 $out\text{-}var\ x \bowtie in\text{-}var\ y$   
**by** (simp add: lens-indep-def in-var-def out-var-def fst-lens-def snd-lens-def lens-comp-def)

**lemma** *in-var-indep* [simp]:  
 $x \bowtie y \implies in\text{-}var\ x \bowtie in\text{-}var\ y$   
**by** (simp add: in-var-def out-var-def)

**lemma** *out-var-indep* [simp]:  
 $x \bowtie y \implies out\text{-}var\ x \bowtie out\text{-}var\ y$   
**by** (simp add: out-var-def)

**lemma** *pr-var-indeps* [simp]:  
 $x \bowtie y \implies pr\text{-}var\ x \bowtie y$   
 $x \bowtie y \implies x \bowtie pr\text{-}var\ y$   
**by** (simp-all add: pr-var-def)

**lemma** *prod-lens-indep-in-var* [simp]:  
 $a \bowtie x \implies a \times_L b \bowtie in\text{-}var\ x$   
**by** (metis in-var-def in-var-indep out-in-indep out-var-def plus-pres-lens-indep prod-as-plus)

**lemma** *prod-lens-indep-out-var* [simp]:  
 $b \bowtie x \implies a \times_L b \bowtie out\text{-}var\ x$   
**by** (metis in-out-indep in-var-def out-var-def out-var-indep plus-pres-lens-indep prod-as-plus)

**lemma** *in-var-pr-var* [simp]:  
 $in\text{-}var\ (pr\text{-}var\ x) = in\text{-}var\ x$   
**by** (simp add: pr-var-def)

**lemma** *out-var-pr-var* [simp]:  
 $out\text{-}var\ (pr\text{-}var\ x) = out\text{-}var\ x$   
**by** (simp add: pr-var-def)

**lemma** *pr-var-idem* [simp]:  
 $pr\text{-}var\ (pr\text{-}var\ x) = pr\text{-}var\ x$   
**by** (simp add: pr-var-def)

**lemma** *pr-var-lens-plus* [simp]:  
 $pr\text{-}var\ (x +_L y) = (x +_L y)$   
**by** (simp add: pr-var-def)

**lemma** *pr-var-lens-comp-1* [simp]:  
 $pr\text{-}var\ x ;_L y = pr\text{-}var\ (x ;_L y)$   
**by** (simp add: pr-var-def)

**lemma** *in-var-plus* [simp]:  $in\text{-}var\ (x +_L y) = in\text{-}var\ x +_L in\text{-}var\ y$   
**by** (simp add: in-var-def)

**lemma** *out-var-plus* [simp]:  $out\text{-}var\ (x +_L y) = out\text{-}var\ x +_L out\text{-}var\ y$   
**by** (simp add: out-var-def)

Similar properties follow for sublens

**lemma** *in-var-sublens* [simp]:  
 $y \subseteq_L x \implies in\text{-}var\ y \subseteq_L in\text{-}var\ x$   
**by** (metis (no-types, hide-lams) in-var-def lens-comp-assoc sublens-def)

**lemma** *out-var-sublens* [simp]:  
 $y \subseteq_L x \implies out\text{-}var\ y \subseteq_L out\text{-}var\ x$   
**by** (metis (no-types, hide-lams) out-var-def lens-comp-assoc sublens-def)

**lemma** *pr-var-sublens* [simp]:  
 $y \subseteq_L x \implies pr\text{-}var\ y \subseteq_L pr\text{-}var\ x$   
**by** (simp add: pr-var-def)

## 2.4 Lens simplifications

We also define some lookup abstraction simplifications.

**lemma** *var-lookup-in* [simp]:  $lens\text{-}get\ (in\text{-}var\ x)\ (A, A') = lens\text{-}get\ x\ A$   
**by** (simp add: in-var-def fst-lens-def lens-comp-def)

**lemma** *var-lookup-out* [simp]:  $lens\text{-}get\ (out\text{-}var\ x)\ (A, A') = lens\text{-}get\ x\ A'$   
**by** (simp add: out-var-def snd-lens-def lens-comp-def)

**lemma** *var-update-in* [simp]:  $lens\text{-}put\ (in\text{-}var\ x)\ (A, A')\ v = (lens\text{-}put\ x\ A\ v, A')$   
**by** (simp add: in-var-def fst-lens-def lens-comp-def)

**lemma** *var-update-out* [simp]:  $lens\text{-}put\ (out\text{-}var\ x)\ (A, A')\ v = (A, lens\text{-}put\ x\ A'\ v)$   
**by** (simp add: out-var-def snd-lens-def lens-comp-def)

**lemma** *get-lens-plus* [simp]:  $get_x +_L y\ s = (get_x\ s, get_y\ s)$   
**by** (simp add: lens-defs)

## 2.5 Syntax translations

In order to support nice syntax for variables, we here set up some translations. The first step is to introduce a collection of non-terminals.

**nonterminal** *svid* and *svids* and *svar* and *svars* and *salpha*

These non-terminals correspond to the following syntactic entities. Non-terminal *svid* is an atomic variable identifier, and *svids* is a list of identifier. *svar* is a decorated variable, such as an input or output variable, and *svars* is a list of decorated variables. *salpha* is an alphabet or set of variables. Such sets can be constructed only through lens composition due to typing restrictions. Next we introduce some syntax constructors.

**syntax** — Identifiers

```
-svid      :: id ⇒ svid (- [999] 999)
-svid-unit  :: svid ⇒ svids (-)
-svid-list  :: svid ⇒ svids ⇒ svids (-,/ -)
-svid-alpha :: svid (v)
-svid-dot   :: svid ⇒ svid ⇒ svid (-:- [998,999] 998)
-mk-svid-list :: svids ⇒ logic — Helper function for summing a list of identifiers
```

A variable identifier can either be a HOL identifier, the complete set of variables in the alphabet **v**, or a composite identifier separated by colons, which corresponds to a sort of qualification. The final option is effectively a lens composition.

**syntax** — Decorations

```
-sivar      :: svid ⇒ svar (&- [990] 990)
-sinvar     :: svid ⇒ svar ($- [990] 990)
-soutvar    :: svid ⇒ svar ($-' [990] 990)
```

A variable can be decorated with an ampersand, to indicate it is a predicate variable, with a dollar to indicate its an unprimed relational variable, or a dollar and “acute” symbol to indicate its a primed relational variable. Isabelle’s parser is extensible so additional decorations can be and are added later.

**syntax** — Variable sets

```
-salphaid   :: svid ⇒ salpha (- [990] 990)
-salphavar  :: svar ⇒ salpha (- [990] 990)
-salphaparen :: salpha ⇒ salpha ('(-'))
-salphacomp :: salpha ⇒ salpha ⇒ salpha (infixr ; 75)
-salphaprod :: salpha ⇒ salpha ⇒ salpha (infixr × 85)
-salphi-all :: salpha (Σ)
-salphi-none :: salpha (∅)
-svar-nil   :: svar ⇒ svars (-)
-svar-cons  :: svar ⇒ svars ⇒ svars (-,/ -)
-salphaset  :: svars ⇒ salpha ({-})
-salphamk   :: logic ⇒ salpha
```

The terminals of an alphabet are either HOL identifiers or UTP variable identifiers. We support two ways of constructing alphabets; by composition of smaller alphabets using a semi-colon or by a set-style construction  $\{a, b, c\}$  with a list of UTP variables.

**syntax** — Quotations

```
-ualpha-set :: svars ⇒ logic ({-})α
-svar      :: svar ⇒ logic ('(-'))v
```

For various reasons, the syntax constructors above all yield specific grammar categories and will not parse at the HOL top level (basically this is to do with us wanting to reuse the syntax for expressions). As a result we provide some quotation constructors above.

Next we need to construct the syntax translations rules. First we need a few polymorphic constants.

#### consts

$svar :: 'v \Rightarrow 'e$   
 $ivar :: 'v \Rightarrow 'e$   
 $ovar :: 'v \Rightarrow 'e$

#### ad hoc-overloading

$svar$  *pr-var* **and**  $ivar$  *in-var* **and**  $ovar$  *out-var*

The functions above turn a representation of a variable (type  $'v$ ), including its name and type, into some lens type  $'e$ .  $svar$  constructs a predicate variable,  $ivar$  and input variables, and  $ovar$  and output variable. The functions bridge between the model and encoding of the variable and its interpretation as a lens in order to integrate it into the general lens-based framework. Overriding these functions is then all we need to make use of any kind of variables in terms of interfacing it with the system. Although in core UTP variables are always modelled using record field, we can overload these constants to allow other kinds of variables, such as deep variables with explicit syntax and type information.

Finally, we set up the translations rules.

#### translations

— Identifiers

$-svid\ x \rightarrow x$   
 $-svid-alpha \Rightarrow \Sigma$   
 $-svid-dot\ x\ y \rightarrow y ;_L x$   
 $-mk-svid-list\ (-svid-unit\ x) \rightarrow x$   
 $-mk-svid-list\ (-svid-list\ xs) \rightarrow x +_L -mk-svid-list\ xs$

— Decorations

$-spvar\ \Sigma \leftarrow CONST\ svar\ CONST\ id-lens$   
 $-sinvar\ \Sigma \leftarrow CONST\ ivar\ 1_L$   
 $-soutvar\ \Sigma \leftarrow CONST\ ovar\ 1_L$   
 $-spvar\ (-svid-dot\ x\ y) \leftarrow CONST\ svar\ (CONST\ lens-comp\ y\ x)$   
 $-sinvar\ (-svid-dot\ x\ y) \leftarrow CONST\ ivar\ (CONST\ lens-comp\ y\ x)$   
 $-soutvar\ (-svid-dot\ x\ y) \leftarrow CONST\ ovar\ (CONST\ lens-comp\ y\ x)$   
 $-svid-dot\ (-svid-dot\ x\ y)\ z \leftarrow -svid-dot\ (CONST\ lens-comp\ y\ x)\ z$

$-spvar\ x \Rightarrow CONST\ svar\ x$   
 $-sinvar\ x \Rightarrow CONST\ ivar\ x$   
 $-soutvar\ x \Rightarrow CONST\ ovar\ x$

— Alphabets

$-salphaparen\ a \rightarrow a$   
 $-salphaid\ x \rightarrow x$   
 $-salphacomp\ x\ y \rightarrow x +_L y$   
 $-salphaprod\ a\ b \Rightarrow a \times_L b$   
 $-salphavar\ x \rightarrow x$   
 $-svar-nil\ x \rightarrow x$   
 $-svar-cons\ x\ xs \rightarrow x +_L xs$   
 $-salphaset\ A \rightarrow A$   
 $(-svar-cons\ x\ (-salphamk\ y)) \leftarrow -salphamk\ (x +_L y)$   
 $x \leftarrow -salphamk\ x$   
 $-salpha-all \Rightarrow 1_L$   
 $-salpha-none \Rightarrow 0_L$

— Quotations  
*-ualpha-set*  $A \rightarrow A$   
*-svar*  $x \rightarrow x$

The translation rules mainly convert syntax into lens constructions, using a mixture of lens operators and the bespoke variable definitions. Notably, a colon variable identifier qualification becomes a lens composition, and variable sets are constructed using `len sum`. The translation rules are carefully crafted to ensure both parsing and pretty printing.

Finally we create the following useful utility translation function that allows us to construct a UTP variable (lens) type given a return and alphabet type.

**syntax**

*-uvar-ty*  $:: \text{type} \Rightarrow \text{type} \Rightarrow \text{type}$

**parse-translation**  $\langle$

*let*

*fun uvar-ty-tr [ty] = Syntax.const @{type-syntax lens} \$ ty \$ Syntax.const @{type-syntax dummy}*  
*| uvar-ty-tr ts = raise TERM (uvar-ty-tr, ts);*

*in [(@{syntax-const -uvar-ty}, K uvar-ty-tr)] end*

*⟩*

**end**

### 3 UTP Expressions

**theory** *utp-expr*

**imports**

*utp-var*

**begin**

#### 3.1 Expression type

**purge-notation** *BNF-Def.convolve*  $((-, / -))$

Before building the predicate model, we will build a model of expressions that generalise alphabetised predicates. Expressions are represented semantically as mapping from the alphabet  $'\alpha$  to the expression's type  $'a$ . This general model will allow us to unify all constructions under one type. The majority definitions in the file are given using the *lifting* package [23], which allows us to reuse much of the existing library of HOL functions.

**typedef**  $(t, 'a) \text{ uexpr} = \text{UNIV} :: ('a \Rightarrow t) \text{ set} ..$

**setup-lifting** *type-definition-uexpr*

**notation** *Rep-uexpr*  $(\llbracket - \rrbracket_e)$

**notation** *Abs-uexpr*  $(mk_e)$

**lemma** *uexpr-eq-iff*:

$e = f \longleftrightarrow (\forall b. \llbracket e \rrbracket_e b = \llbracket f \rrbracket_e b)$

**using** *Rep-uexpr-inject*  $[of\ e\ f, THEN\ sym]$  **by** *(auto)*

The term  $\llbracket e \rrbracket_e b$  effectively refers to the semantic interpretation of the expression under the state-space valuation (or variables binding)  $b$ . It can be used, in concert with the lifting package, to interpret UTP constructs to their HOL equivalents. We create some theorem sets to store such transfer theorems.

named-theorems *uepr-defs* and *ueval* and *lit-simps* and *lit-norm*

### 3.2 Core expression constructs

A variable expression corresponds to the lens *get* function associated with a variable. Specifically, given a lens the expression always returns that portion of the state-space referred to by the lens.

**lift-definition**  $var :: ('t \Rightarrow 'a) \Rightarrow ('t, 'a) \text{ uepr is lens-get } .$

A literal is simply a constant function expression, always returning the same value for any binding.

**lift-definition**  $lit :: 't \Rightarrow ('t, 'a) \text{ uepr } (\ll-\gg) \text{ is } \lambda v b. v .$

We define lifting for unary, binary, ternary, and quaternary expression constructs, that simply take a HOL function with correct number of arguments and apply it function to all possible results of the expressions.

**lift-definition**  $uop :: ('a \Rightarrow 'b) \Rightarrow ('a, 'a) \text{ uepr } \Rightarrow ('b, 'a) \text{ uepr}$   
**is**  $\lambda f e b. f (e b) .$

**lift-definition**  $bop ::$   
 $('a \Rightarrow 'b \Rightarrow 'c) \Rightarrow ('a, 'a) \text{ uepr } \Rightarrow ('b, 'a) \text{ uepr } \Rightarrow ('c, 'a) \text{ uepr}$   
**is**  $\lambda f u v b. f (u b) (v b) .$

**lift-definition**  $trop ::$   
 $('a \Rightarrow 'b \Rightarrow 'c \Rightarrow 'd) \Rightarrow ('a, 'a) \text{ uepr } \Rightarrow ('b, 'a) \text{ uepr } \Rightarrow ('c, 'a) \text{ uepr } \Rightarrow ('d, 'a) \text{ uepr}$   
**is**  $\lambda f u v w b. f (u b) (v b) (w b) .$

**lift-definition**  $qtop ::$   
 $('a \Rightarrow 'b \Rightarrow 'c \Rightarrow 'd \Rightarrow 'e) \Rightarrow$   
 $('a, 'a) \text{ uepr } \Rightarrow ('b, 'a) \text{ uepr } \Rightarrow ('c, 'a) \text{ uepr } \Rightarrow ('d, 'a) \text{ uepr } \Rightarrow$   
 $('e, 'a) \text{ uepr}$   
**is**  $\lambda f u v w x b. f (u b) (v b) (w b) (x b) .$

We also define a UTP expression version of function ( $\lambda$ ) abstraction, that takes a function producing an expression and produces an expression producing a function.

**lift-definition**  $ulambda :: ('a \Rightarrow ('b, 'a) \text{ uepr}) \Rightarrow ('a \Rightarrow 'b, 'a) \text{ uepr}$   
**is**  $\lambda f A x. f x A .$

We set up syntax for the conditional. This is effectively an infix version of if-then-else where the condition is in the middle.

**definition**  $uIf :: bool \Rightarrow 'a \Rightarrow 'a \Rightarrow 'a \text{ where}$   
 $[uepr-defs]: uIf = If$

**abbreviation**  $cond ::$   
 $('a, 'a) \text{ uepr } \Rightarrow (bool, 'a) \text{ uepr } \Rightarrow ('a, 'a) \text{ uepr } \Rightarrow ('a, 'a) \text{ uepr}$   
 $((\beta - \triangleleft - \triangleright / -) [52, 0, 53] 52)$   
**where**  $P \triangleleft b \triangleright Q \equiv trop uIf b P Q$

UTP expression is equality is simply HOL equality lifted using the *bop* binary expression constructor.

**definition**  $eq-upred :: ('a, 'a) \text{ uepr } \Rightarrow ('a, 'a) \text{ uepr } \Rightarrow (bool, 'a) \text{ uepr } (\text{infixl } =_u 50)$   
**where**  $[uepr-defs]: eq-upred x y = bop HOL.eq x y$

A literal is the expression  $\ll v \gg$ , where  $v$  is any HOL term. Actually, the literal construct is very versatile and also allows us to refer to HOL variables within UTP expressions, and has a variety of other uses. It can therefore also be considered as a kind of quotation mechanism.



We also set up syntax for UTP variable expressions.

**syntax**

*-uuvar* :: *svar*  $\Rightarrow$  *logic* (-)

**translations**

*-uuvar* *x* == *CONST* *var* *x*

Since we already have a parser for variables, we can directly reuse it and simply apply the *var* expression construct to lift the resulting variable to an expression.

### 3.3 Type class instantiations

Isabelle/HOL of course provides a large hierarchy of type classes that provide constructs such as numerals and the arithmetic operators. Fortunately we can directly make use of these for UTP expressions, and thus we now perform a long list of appropriate instantiations. We first lift the core arithmetic constants and operators using a mixture of literals, unary, and binary expression constructors.

**instantiation** *uexpr* :: (*zero*, *type*) *zero*

**begin**

**definition** *zero-uexpr-def* [*uexpr-defs*]: *0* = *lit 0*

**instance** ..

**end**

**instantiation** *uexpr* :: (*one*, *type*) *one*

**begin**

**definition** *one-uexpr-def* [*uexpr-defs*]: *1* = *lit 1*

**instance** ..

**end**

**instantiation** *uexpr* :: (*plus*, *type*) *plus*

**begin**

**definition** *plus-uexpr-def* [*uexpr-defs*]: *u* + *v* = *bop* (+) *u* *v*

**instance** ..

**end**

**instance** *uexpr* :: (*semigroup-add*, *type*) *semigroup-add*

**by** (*intro-classes*) (*simp add: plus-uexpr-def zero-uexpr-def*, *transfer*, *simp add: add.assoc*)

The following instantiation sets up numerals. This will allow us to have Isabelle number representations (i.e. 3,7,42,198 etc.) to UTP expressions directly.

**instance** *uexpr* :: (*numeral*, *type*) *numeral*

**by** (*intro-classes*, *simp add: plus-uexpr-def*, *transfer*, *simp add: add.assoc*)

We can also define the order relation on expressions. Now, unlike the previous group and ring constructs, the order relations ( $\leq$ ) and ( $\leq$ ) return a *bool* type. This order is not therefore the lifted order which allows us to compare the valuation of two expressions, but rather the order on expressions themselves. Notably, this instantiation will later allow us to talk about predicate refinements and complete lattices.

**instantiation** *uexpr* :: (*ord*, *type*) *ord*

**begin**

**lift-definition** *less-eq-uexpr* :: ('*a*, '*b*) *uexpr*  $\Rightarrow$  ('*a*, '*b*) *uexpr*  $\Rightarrow$  *bool*

```

is  $\lambda P Q. (\forall A. P A \leq Q A) .$ 
definition less-uepr :: ('a, 'b) uepr  $\Rightarrow$  ('a, 'b) uepr  $\Rightarrow$  bool
where [uepr-defs]: less-uepr P Q = (P  $\leq$  Q  $\wedge$   $\neg$  Q  $\leq$  P)
instance ..
end

```

UTP expressions whose return type is a partial ordered type, are also partially ordered as the following instantiation demonstrates.

```

instance uepr :: (order, type) order
proof
  fix x y z :: ('a, 'b) uepr
  show (x < y) = (x  $\leq$  y  $\wedge$   $\neg$  y  $\leq$  x) by (simp add: less-uepr-def)
  show x  $\leq$  x by (transfer, auto)
  show x  $\leq$  y  $\Longrightarrow$  y  $\leq$  z  $\Longrightarrow$  x  $\leq$  z
    by (transfer, blast intro: order.trans)
  show x  $\leq$  y  $\Longrightarrow$  y  $\leq$  x  $\Longrightarrow$  x = y
    by (transfer, rule ext, simp add: eq-iff)
qed

```

### 3.4 Syntax translations

The follows a large number of translations that lift HOL functions to UTP expressions using the various expression constructors defined above. Much of the time we try to keep the HOL syntax but add a "u" subscript.

**abbreviation** (*input*) *ulens-override* x f g  $\equiv$  *lens-override* f g x

This operator allows us to get the characteristic set of a type. Essentially this is *UNIV*, but it retains the type syntactically for pretty printing.

**definition** *set-of* :: 'a itself  $\Rightarrow$  'a set **where**  
 [*uepr-defs*]: *set-of* t = *UNIV*

We add new non-terminals for UTP tuples and maplets.

**nonterminal** *utuple-args* **and** *umaplet* **and** *umaplets*

**syntax** — Core expression constructs

```

-ucorcer :: logic  $\Rightarrow$  type  $\Rightarrow$  logic (infix :u 50)
-ulambda :: ptrn  $\Rightarrow$  logic  $\Rightarrow$  logic ( $\lambda$  - - - [0, 10] 10)
-ulens-ovrd :: logic  $\Rightarrow$  logic  $\Rightarrow$  salpha  $\Rightarrow$  logic (-  $\oplus$  - on - [85, 0, 86] 86)
-ulens-get :: logic  $\Rightarrow$  svar  $\Rightarrow$  logic (-:- [900, 901] 901)
-umem :: ('a, 'α) uepr  $\Rightarrow$  ('a set, 'α) uepr  $\Rightarrow$  (bool, 'α) uepr (infix  $\in$ u 50)

```

**translations**

```

 $\lambda x \cdot p ==$  CONST ulambda ( $\lambda x. p$ )
x :u 'a == x :: ('a, -) uepr
-ulens-ovrd f g a => CONST bop (CONST ulens-override a) f g
-ulens-ovrd f g a <= CONST bop ( $\lambda x y. \text{CONST lens-override } x1 y1 a$ ) f g
-ulens-get x y == CONST uop (CONST lens-get y) x
x  $\in$ u A == CONST bop ( $\in$ ) x A

```

**syntax** — Tuples

```

-utuple :: ('a, 'α) uepr  $\Rightarrow$  utuple-args  $\Rightarrow$  ('a * 'b, 'α) uepr ((1'(-, / -)u))
-utuple-arg :: ('a, 'α) uepr  $\Rightarrow$  utuple-args (-)
-utuple-args :: ('a, 'α) uepr => utuple-args  $\Rightarrow$  utuple-args (-, / -)
-uunit :: ('a, 'α) uepr ('()u)

```

$-ufst \quad :: ('a \times 'b, 'a) \text{ uepr} \Rightarrow ('a, 'a) \text{ uepr} (\pi_1 '(-))$   
 $-usnd \quad :: ('a \times 'b, 'a) \text{ uepr} \Rightarrow ('b, 'a) \text{ uepr} (\pi_2 '(-))$

#### translations

$()_u \quad == \langle\langle() \rangle\rangle$   
 $(x, y)_u \quad == \text{CONST bop} (\text{CONST Pair}) x y$   
 $\text{-utuple } x \text{ (-utuple-args } y \text{ } z) \quad == \text{-utuple } x \text{ (-utuple-arg } (-\text{utuple } y \text{ } z))$   
 $\pi_1(x) \quad == \text{CONST uop} \text{CONST fst } x$   
 $\pi_2(x) \quad == \text{CONST uop} \text{CONST snd } x$

#### syntax — Orders

$\text{-ules} \quad :: \text{logic} \Rightarrow \text{logic} \Rightarrow \text{logic} \text{ (infix } <_u \text{ } 50)$   
 $\text{-uleq} \quad :: \text{logic} \Rightarrow \text{logic} \Rightarrow \text{logic} \text{ (infix } \leq_u \text{ } 50)$   
 $\text{-ugreat} \quad :: \text{logic} \Rightarrow \text{logic} \Rightarrow \text{logic} \text{ (infix } >_u \text{ } 50)$   
 $\text{-ugeq} \quad :: \text{logic} \Rightarrow \text{logic} \Rightarrow \text{logic} \text{ (infix } \geq_u \text{ } 50)$

#### translations

$x <_u y \quad == \text{CONST bop} (<) x y$   
 $x \leq_u y \quad == \text{CONST bop} (\leq) x y$   
 $x >_u y \quad ==> y <_u x$   
 $x \geq_u y \quad ==> y \leq_u x$

### 3.5 Evaluation laws for expressions

The following laws show how to evaluate the core expressions constructs in terms of which the above definitions are defined. Thus, using these theorems together, we can convert any UTP expression into a pure HOL expression. All these theorems are marked as *ueval* theorems which can be used for evaluation.

**lemma** *lit-ueval* [ueval]:  $\llbracket \langle\langle x \rangle\rangle \rrbracket_e b = x$   
**by** (*transfer*, *simp*)

**lemma** *var-ueval* [ueval]:  $\llbracket \text{var } x \rrbracket_e b = \text{get}_x b$   
**by** (*transfer*, *simp*)

**lemma** *uop-ueval* [ueval]:  $\llbracket \text{uop } f \text{ } x \rrbracket_e b = f (\llbracket x \rrbracket_e b)$   
**by** (*transfer*, *simp*)

**lemma** *bop-ueval* [ueval]:  $\llbracket \text{bop } f \text{ } x \text{ } y \rrbracket_e b = f (\llbracket x \rrbracket_e b) (\llbracket y \rrbracket_e b)$   
**by** (*transfer*, *simp*)

**lemma** *trop-ueval* [ueval]:  $\llbracket \text{trop } f \text{ } x \text{ } y \text{ } z \rrbracket_e b = f (\llbracket x \rrbracket_e b) (\llbracket y \rrbracket_e b) (\llbracket z \rrbracket_e b)$   
**by** (*transfer*, *simp*)

**lemma** *qtop-ueval* [ueval]:  $\llbracket \text{qtop } f \text{ } x \text{ } y \text{ } z \text{ } w \rrbracket_e b = f (\llbracket x \rrbracket_e b) (\llbracket y \rrbracket_e b) (\llbracket z \rrbracket_e b) (\llbracket w \rrbracket_e b)$   
**by** (*transfer*, *simp*)

### 3.6 Misc laws

We also prove a few useful algebraic and expansion laws for expressions.

**lemma** *uop-const* [simp]:  $\text{uop id } u = u$   
**by** (*transfer*, *simp*)

**lemma** *bop-const-1* [simp]:  $\text{bop } (\lambda x \text{ } y. \text{ } y) u v = v$   
**by** (*transfer*, *simp*)

**lemma** *bop-const-2* [*simp*]:  $bop (\lambda x y. x) u v = u$   
**by** (*transfer*, *simp*)

**lemma** *uexpr-fst* [*simp*]:  $\pi_1((e, f)_u) = e$   
**by** (*transfer*, *simp*)

**lemma** *uexpr-snd* [*simp*]:  $\pi_2((e, f)_u) = f$   
**by** (*transfer*, *simp*)

### 3.7 Literalise tactics

The following tactic converts literal HOL expressions to UTP expressions and vice-versa via a collection of simplification rules. The two tactics are called "literalise", which converts UTP to expressions to HOL expressions – i.e. it pushes them into literals – and *unliteralise* that reverses this. We collect the equations in a theorem attribute called "lit\_simps".

**lemma** *lit-fun-simps* [*lit\_simps*]:  
 $\llbracket i \ x \ y \ z \ u \rrbracket = qtop \ i \ \llbracket x \rrbracket \ \llbracket y \rrbracket \ \llbracket z \rrbracket \ \llbracket u \rrbracket$   
 $\llbracket h \ x \ y \ z \rrbracket = trop \ h \ \llbracket x \rrbracket \ \llbracket y \rrbracket \ \llbracket z \rrbracket$   
 $\llbracket g \ x \ y \rrbracket = bop \ g \ \llbracket x \rrbracket \ \llbracket y \rrbracket$   
 $\llbracket f \ x \rrbracket = uop \ f \ \llbracket x \rrbracket$   
**by** (*transfer*, *simp*)<sup>+</sup>

The following two theorems also set up interpretation of numerals, meaning a UTP numeral can always be converted to a HOL numeral.

**lemma** *numeral-uexpr-rep-eq* [*ueval*]:  $\llbracket numeral \ x \rrbracket_e \ b = numeral \ x$   
**apply** (*induct* *x*)  
**apply** (*simp* *add*: *lit.rep-eq one-uexpr-def*)  
**apply** (*simp* *add*: *bop.rep-eq numeral-Bit0 plus-uexpr-def*)  
**apply** (*simp* *add*: *bop.rep-eq lit.rep-eq numeral-code(3) one-uexpr-def plus-uexpr-def*)  
**done**

**lemma** *numeral-uexpr-simp*:  $numeral \ x = \llbracket numeral \ x \rrbracket$   
**by** (*simp* *add*: *uexpr-eq-iff numeral-uexpr-rep-eq lit.rep-eq*)

**lemma** *lit-zero* [*lit\_simps*]:  $\llbracket 0 \rrbracket = 0$  **by** (*simp* *add*: *uexpr-defs*)

**lemma** *lit-one* [*lit\_simps*]:  $\llbracket 1 \rrbracket = 1$  **by** (*simp* *add*: *uexpr-defs*)

**lemma** *lit-plus* [*lit\_simps*]:  $\llbracket x + y \rrbracket = \llbracket x \rrbracket + \llbracket y \rrbracket$  **by** (*simp* *add*: *uexpr-defs*, *transfer*, *simp*)

**lemma** *lit-numeral* [*lit\_simps*]:  $\llbracket numeral \ n \rrbracket = numeral \ n$  **by** (*simp* *add*: *numeral-uexpr-simp*)

In general *unliteralising* converts function applications to corresponding expression liftings.

Since some operators, like + and \*, have specific operators we also have to use  $uIf = If$

$(?x =_u ?y) = bop (=) ?x ?y$

$0 = \llbracket 0 \rrbracket :: ?'a$

$1 = \llbracket 1 \rrbracket :: ?'a$

$?u + ?v = bop (+) ?u ?v$

$(?P < ?Q) = (?P \leq ?Q \wedge \neg ?Q \leq ?P)$

*set-of*  $?t = UNIV$  in reverse to correctly interpret these. Moreover, numerals must be handled separately by first simplifying them and then converting them into UTP expression numerals; hence the following two simplification rules.

**lemma** *lit-numeral-1*:  $uop \ numeral \ x = Abs-uexpr (\lambda b. numeral (\llbracket x \rrbracket_e \ b))$   
**by** (*simp* *add*: *uop-def*)

**lemma** *lit-numeral-2*:  $Abs-uepr (\lambda b. numeral\ v) = numeral\ v$   
**by** (*metis lit.abs-eq lit-numeral*)

**method** *literalise* = (*unfold lit-simps [THEN sym]*)  
**method** *unliteralise* = (*unfold lit-simps uepr-defs [THEN sym];*  
*(unfold lit-numeral-1 ; (unfold uepr-defs ueval); (unfold lit-numeral-2))?*) +

The following tactic can be used to evaluate literal expressions. It first literalises UTP expressions, that is pushes as many operators into literals as possible. Then it tries to simplify, and final unliteralises at the end.

**method** *uepr-simp* **uses** *simps* = ((*literalise*)?, *simp add: lit-norm simps, (unliteralise)?*)

**lemma**  $(1::(int, 'a) uepr) + \ll 2 \gg = 4 \longleftrightarrow \ll 3 \gg = 4$   
**apply** (*literalise*)  
**apply** (*uepr-simp*) **oops**

**end**

## 4 Expression Type Class Instantiations

**theory** *utp-expr-insts*  
**imports** *utp-expr*  
**begin**

It should be noted that instantiating the unary minus class, *uminus*, will also provide negation UTP predicates later.

**instantiation** *uepr* :: (*uminus, type*) *uminus*  
**begin**  
**definition** *uminus-uepr-def* [*uepr-defs*]:  $- u = uop\ uminus\ u$   
**instance** ..  
**end**

**instantiation** *uepr* :: (*minus, type*) *minus*  
**begin**  
**definition** *minus-uepr-def* [*uepr-defs*]:  $u - v = bop\ (-)\ u\ v$   
**instance** ..  
**end**

**instantiation** *uepr* :: (*times, type*) *times*  
**begin**  
**definition** *times-uepr-def* [*uepr-defs*]:  $u * v = bop\ times\ u\ v$   
**instance** ..  
**end**

**instance** *uepr* :: (*Rings.dvd, type*) *Rings.dvd* ..

**instantiation** *uepr* :: (*divide, type*) *divide*  
**begin**  
**definition** *divide-uepr* :: (*'a, 'b*) *uepr*  $\Rightarrow$  (*'a, 'b*) *uepr*  $\Rightarrow$  (*'a, 'b*) *uepr* **where**  
*[uepr-defs]: divide-uepr u v = bop divide u v*  
**instance** ..

**end**

**instantiation** *uexpr* :: (*inverse*, *type*) *inverse*

**begin**

**definition** *inverse-uexpr* :: ('a, 'b) *uexpr*  $\Rightarrow$  ('a, 'b) *uexpr*

**where** [*uexpr-defs*]: *inverse-uexpr* u = *uop inverse* u

**instance** ..

**end**

**instantiation** *uexpr* :: (*modulo*, *type*) *modulo*

**begin**

**definition** *mod-uexpr-def* [*uexpr-defs*]: *u mod v* = *bop (mod)* u *v*

**instance** ..

**end**

**instantiation** *uexpr* :: (*sgn*, *type*) *sgn*

**begin**

**definition** *sgn-uexpr-def* [*uexpr-defs*]: *sgn u* = *uop sgn* u

**instance** ..

**end**

**instantiation** *uexpr* :: (*abs*, *type*) *abs*

**begin**

**definition** *abs-uexpr-def* [*uexpr-defs*]: *abs u* = *uop abs* u

**instance** ..

**end**

Once we've set up all the core constructs for arithmetic, we can also instantiate the type classes for various algebras, including groups and rings. The proofs are done by definitional expansion, the *transfer* tactic, and then finally the theorems of the underlying HOL operators. This is mainly routine, so we don't comment further.

**instance** *uexpr* :: (*semigroup-mult*, *type*) *semigroup-mult*

**by** (*intro-classes*) (*simp add: times-uexpr-def one-uexpr-def, transfer, simp add: mult.assoc*) +

**instance** *uexpr* :: (*monoid-mult*, *type*) *monoid-mult*

**by** (*intro-classes*) (*simp add: times-uexpr-def one-uexpr-def, transfer, simp*) +

**instance** *uexpr* :: (*monoid-add*, *type*) *monoid-add*

**by** (*intro-classes*) (*simp add: plus-uexpr-def zero-uexpr-def, transfer, simp*) +

**instance** *uexpr* :: (*ab-semigroup-add*, *type*) *ab-semigroup-add*

**by** (*intro-classes*) (*simp add: plus-uexpr-def, transfer, simp add: add.commute*) +

**instance** *uexpr* :: (*cancel-semigroup-add*, *type*) *cancel-semigroup-add*

**by** (*intro-classes*) (*simp add: plus-uexpr-def, transfer, simp add: fun-eq-iff*) +

**instance** *uexpr* :: (*cancel-ab-semigroup-add*, *type*) *cancel-ab-semigroup-add*

**by** (*intro-classes*, (*simp add: plus-uexpr-def minus-uexpr-def, transfer, simp add: fun-eq-iff add.commute cancel-ab-semigroup-add-class.diff-diff-add*) +)

**instance** *uexpr* :: (*group-add*, *type*) *group-add*

**by** (*intro-classes*)

    (*simp add: plus-uexpr-def uminus-uexpr-def minus-uexpr-def zero-uexpr-def, transfer, simp*) +

**instance** *uexpr* :: (*ab-group-add*, *type*) *ab-group-add*

**by** (*intro-classes*)  
 (*simp add: plus-uepr-def uminus-uepr-def minus-uepr-def zero-uepr-def, transfer, simp*)+

**instance** *uepr* :: (*semiring, type*) *semiring*  
**by** (*intro-classes*) (*simp add: plus-uepr-def times-uepr-def, transfer, simp add: fun-eq-iff add.commute semiring-class.distrib-right semiring-class.distrib-left*)+

**instance** *uepr* :: (*ring-1, type*) *ring-1*  
**by** (*intro-classes*) (*simp add: plus-uepr-def uminus-uepr-def minus-uepr-def times-uepr-def zero-uepr-def one-uepr-def, transfer, simp add: fun-eq-iff*)+

We also lift the properties from certain ordered groups.

**instance** *uepr* :: (*ordered-ab-group-add, type*) *ordered-ab-group-add*  
**by** (*intro-classes*) (*simp add: plus-uepr-def, transfer, simp*)

**instance** *uepr* :: (*ordered-ab-group-add-abs, type*) *ordered-ab-group-add-abs*  
**apply** (*intro-classes*)  
**apply** (*simp add: abs-uepr-def zero-uepr-def plus-uepr-def uminus-uepr-def, transfer, simp add: abs-ge-self abs-le-iff abs-triangle-ineq*) +  
**apply** (*metis ab-group-add-class.ab-diff-conv-add-uminus abs-ge-minus-self abs-ge-self add-mono-thms-linordered-semiring*)  
**done**

The next theorem lifts powers.

**lemma** *power-rep-eq* [*ueval*]:  $\llbracket P \wedge n \rrbracket_e = (\lambda b. \llbracket P \rrbracket_e b \wedge n)$   
**by** (*induct n, simp-all add: lit.rep-eq one-uepr-def bop.rep-eq times-uepr-def*)

**lemma** *of-nat-uepr-rep-eq* [*ueval*]:  $\llbracket \text{of-nat } x \rrbracket_e b = \text{of-nat } x$   
**by** (*induct x, simp-all add: uepr-defs ueval*)

**lemma** *lit-uminus* [*lit-simps*]:  $\llbracket -x \rrbracket = - \llbracket x \rrbracket$  **by** (*simp add: uepr-defs, transfer, simp*)  
**lemma** *lit-minus* [*lit-simps*]:  $\llbracket x - y \rrbracket = \llbracket x \rrbracket - \llbracket y \rrbracket$  **by** (*simp add: uepr-defs, transfer, simp*)  
**lemma** *lit-times* [*lit-simps*]:  $\llbracket x * y \rrbracket = \llbracket x \rrbracket * \llbracket y \rrbracket$  **by** (*simp add: uepr-defs, transfer, simp*)  
**lemma** *lit-divide* [*lit-simps*]:  $\llbracket x / y \rrbracket = \llbracket x \rrbracket / \llbracket y \rrbracket$  **by** (*simp add: uepr-defs, transfer, simp*)  
**lemma** *lit-div* [*lit-simps*]:  $\llbracket x \text{ div } y \rrbracket = \llbracket x \rrbracket \text{ div } \llbracket y \rrbracket$  **by** (*simp add: uepr-defs, transfer, simp*)  
**lemma** *lit-power* [*lit-simps*]:  $\llbracket x \wedge n \rrbracket = \llbracket x \rrbracket \wedge n$  **by** (*simp add: lit.rep-eq power-rep-eq uepr-eq-iff*)

## 4.1 Expression construction from HOL terms

Sometimes it is convenient to cast HOL terms to UTP expressions, and these simplifications automate this process.

**named-theorems** *mkuepr*

**lemma** *mkuepr-lens-get* [*mkuepr*]:  $\text{mk}_e \text{ get } x = \&x$   
**by** (*transfer, simp add: pr-var-def*)

**lemma** *mkuepr-zero* [*mkuepr*]:  $\text{mk}_e (\lambda s. 0) = 0$   
**by** (*simp add: zero-uepr-def, transfer, simp*)

**lemma** *mkuepr-one* [*mkuepr*]:  $\text{mk}_e (\lambda s. 1) = 1$   
**by** (*simp add: one-uepr-def, transfer, simp*)

**lemma** *mkuepr-numeral* [*mkuepr*]:  $\text{mk}_e (\lambda s. \text{numeral } n) = \text{numeral } n$   
**using** *lit-numeral-2* **by** *blast*

**lemma** *mkueexpr-lit* [*mkueexpr*]:  $mk_e (\lambda s. k) = \ll k \gg$   
**by** (*transfer*, *simp*)

**lemma** *mkueexpr-pair* [*mkueexpr*]:  $mk_e (\lambda s. (f s, g s)) = (mk_e f, mk_e g)_u$   
**by** (*transfer*, *simp*)

**lemma** *mkueexpr-plus* [*mkueexpr*]:  $mk_e (\lambda s. f s + g s) = mk_e f + mk_e g$   
**by** (*simp add: plus-ueexpr-def*, *transfer*, *simp*)

**lemma** *mkueexpr-uminus* [*mkueexpr*]:  $mk_e (\lambda s. - f s) = - mk_e f$   
**by** (*simp add: uminus-ueexpr-def*, *transfer*, *simp*)

**lemma** *mkueexpr-minus* [*mkueexpr*]:  $mk_e (\lambda s. f s - g s) = mk_e f - mk_e g$   
**by** (*simp add: minus-ueexpr-def*, *transfer*, *simp*)

**lemma** *mkueexpr-times* [*mkueexpr*]:  $mk_e (\lambda s. f s * g s) = mk_e f * mk_e g$   
**by** (*simp add: times-ueexpr-def*, *transfer*, *simp*)

**lemma** *mkueexpr-divide* [*mkueexpr*]:  $mk_e (\lambda s. f s / g s) = mk_e f / mk_e g$   
**by** (*simp add: divide-ueexpr-def*, *transfer*, *simp*)

**end**

**theory** *utp-expr-funcs*

**imports** *utp-expr-insts*

**begin**

**syntax** — Polymorphic constructs

-*uceil* ::  $logic \Rightarrow logic \ (\lceil \_ \rceil_u)$   
-*ufloor* ::  $logic \Rightarrow logic \ (\lfloor \_ \rfloor_u)$   
-*umin* ::  $logic \Rightarrow logic \Rightarrow logic \ (min_u'(-, -'))$   
-*umax* ::  $logic \Rightarrow logic \Rightarrow logic \ (max_u'(-, -'))$   
-*ugcd* ::  $logic \Rightarrow logic \Rightarrow logic \ (gcd_u'(-, -'))$

**translations**

— Type-class polymorphic constructs

$min_u(x, y) == CONST \text{ bop } (CONST \text{ min}) \ x \ y$   
 $max_u(x, y) == CONST \text{ bop } (CONST \text{ max}) \ x \ y$   
 $gcd_u(x, y) == CONST \text{ bop } (CONST \text{ gcd}) \ x \ y$   
 $\lceil x \rceil_u == CONST \text{ uop } CONST \text{ ceiling } x$   
 $\lfloor x \rfloor_u == CONST \text{ uop } CONST \text{ floor } x$

**syntax** — Lists / Sequences

-*ucons* ::  $logic \Rightarrow logic \Rightarrow logic \ (\text{infixr } \#_u \ 65)$   
-*unil* ::  $('a \text{ list}, 'a) \ uexpr \ (\langle \rangle)$   
-*ulist* ::  $args \Rightarrow ('a \text{ list}, 'a) \ uexpr \ (\langle \langle \_ \rangle \rangle)$   
-*uappend* ::  $('a \text{ list}, 'a) \ uexpr \Rightarrow ('a \text{ list}, 'a) \ uexpr \Rightarrow ('a \text{ list}, 'a) \ uexpr \ (\text{infixr } \hat{\_}_u \ 80)$   
-*udconcat* ::  $logic \Rightarrow logic \Rightarrow logic \ (\text{infixr } \frown_u \ 90)$   
-*ulast* ::  $('a \text{ list}, 'a) \ uexpr \Rightarrow ('a, 'a) \ uexpr \ (last_u'(-))$   
-*ufront* ::  $('a \text{ list}, 'a) \ uexpr \Rightarrow ('a \text{ list}, 'a) \ uexpr \ (front_u'(-))$   
-*uhead* ::  $('a \text{ list}, 'a) \ uexpr \Rightarrow ('a, 'a) \ uexpr \ (head_u'(-))$   
-*utail* ::  $('a \text{ list}, 'a) \ uexpr \Rightarrow ('a \text{ list}, 'a) \ uexpr \ (tail_u'(-))$   
-*utake* ::  $(nat, 'a) \ uexpr \Rightarrow ('a \text{ list}, 'a) \ uexpr \Rightarrow ('a \text{ list}, 'a) \ uexpr \ (take_u'(-, -'))$   
-*udrop* ::  $(nat, 'a) \ uexpr \Rightarrow ('a \text{ list}, 'a) \ uexpr \Rightarrow ('a \text{ list}, 'a) \ uexpr \ (drop_u'(-, -'))$   
-*ufilter* ::  $('a \text{ list}, 'a) \ uexpr \Rightarrow ('a \text{ set}, 'a) \ uexpr \Rightarrow ('a \text{ list}, 'a) \ uexpr \ (\text{infixl } \downarrow_u \ 75)$   
-*uextract* ::  $('a \text{ set}, 'a) \ uexpr \Rightarrow ('a \text{ list}, 'a) \ uexpr \Rightarrow ('a \text{ list}, 'a) \ uexpr \ (\text{infixl } \uparrow_u \ 75)$



$-uelems \quad :: ('a \text{ list}, 'α) \text{ uexpr} \Rightarrow ('a \text{ set}, 'α) \text{ uexpr} \text{ (elems}_u'(-))$   
 $-usorted \quad :: ('a \text{ list}, 'α) \text{ uexpr} \Rightarrow (bool, 'α) \text{ uexpr} \text{ (sorted}_u'(-))$   
 $-udistinct \quad :: ('a \text{ list}, 'α) \text{ uexpr} \Rightarrow (bool, 'α) \text{ uexpr} \text{ (distinct}_u'(-))$   
 $-uupto \quad :: logic \Rightarrow logic \Rightarrow logic \text{ (}\langle \dots \rangle\text{)}$   
 $-uupt \quad :: logic \Rightarrow logic \Rightarrow logic \text{ (}\langle \dots < \rangle\text{)}$   
 $-umap \quad :: logic \Rightarrow logic \Rightarrow logic \text{ (map}_u\text{)}$   
 $-uzip \quad :: logic \Rightarrow logic \Rightarrow logic \text{ (zip}_u\text{)}$

#### translations

$x \#_u ys == CONST \text{ bop } (\#) x ys$   
 $\langle \rangle == \ll \square \gg$   
 $\langle x, xs \rangle == x \#_u \langle xs \rangle$   
 $\langle x \rangle == x \#_u \ll \square \gg$   
 $x \hat{ }_u y == CONST \text{ bop } (@) x y$   
 $A \cap_u B == CONST \text{ bop } (\cap) A B$   
 $last_u(xs) == CONST \text{ uop } CONST \text{ last } xs$   
 $front_u(xs) == CONST \text{ uop } CONST \text{ butlast } xs$   
 $head_u(xs) == CONST \text{ uop } CONST \text{ hd } xs$   
 $tail_u(xs) == CONST \text{ uop } CONST \text{ tl } xs$   
 $drop_u(n, xs) == CONST \text{ bop } CONST \text{ drop } n xs$   
 $take_u(n, xs) == CONST \text{ bop } CONST \text{ take } n xs$   
 $elems_u(xs) == CONST \text{ uop } CONST \text{ set } xs$   
 $sorted_u(xs) == CONST \text{ uop } CONST \text{ sorted } xs$   
 $distinct_u(xs) == CONST \text{ uop } CONST \text{ distinct } xs$   
 $xs \downarrow_u A == CONST \text{ bop } CONST \text{ seq-filter } xs A$   
 $A \uparrow_u xs == CONST \text{ bop } (\uparrow_l) A xs$   
 $\langle n..k \rangle == CONST \text{ bop } CONST \text{ upto } n k$   
 $\langle n..<k \rangle == CONST \text{ bop } CONST \text{ upt } n k$   
 $map_u f xs == CONST \text{ bop } CONST \text{ map } f xs$   
 $zip_u xs ys == CONST \text{ bop } CONST \text{ zip } xs ys$

#### syntax — Sets

$-ufinite \quad :: logic \Rightarrow logic \text{ (finite}_u'(-))$   
 $-uempset \quad :: ('a \text{ set}, 'α) \text{ uexpr } (\{\}_u)$   
 $-uset \quad :: args \Rightarrow (> ('a \text{ set}, 'α) \text{ uexpr } (\{(-)\}_u)$   
 $-uunion \quad :: ('a \text{ set}, 'α) \text{ uexpr} \Rightarrow ('a \text{ set}, 'α) \text{ uexpr} \Rightarrow ('a \text{ set}, 'α) \text{ uexpr} \text{ (infixl } \cup_u \text{ 65)}$   
 $-uinter \quad :: ('a \text{ set}, 'α) \text{ uexpr} \Rightarrow ('a \text{ set}, 'α) \text{ uexpr} \Rightarrow ('a \text{ set}, 'α) \text{ uexpr} \text{ (infixl } \cap_u \text{ 70)}$   
 $-uinsert \quad :: logic \Rightarrow logic \Rightarrow logic \text{ (insert}_u\text{)}$   
 $-uimage \quad :: logic \Rightarrow logic \Rightarrow logic \text{ (-\[\_]_u [10,0] 10)}$   
 $-usubset \quad :: ('a \text{ set}, 'α) \text{ uexpr} \Rightarrow ('a \text{ set}, 'α) \text{ uexpr} \Rightarrow (bool, 'α) \text{ uexpr} \text{ (infix } \subseteq_u \text{ 50)}$   
 $-usubseteq \quad :: ('a \text{ set}, 'α) \text{ uexpr} \Rightarrow ('a \text{ set}, 'α) \text{ uexpr} \Rightarrow (bool, 'α) \text{ uexpr} \text{ (infix } \subseteq_u \text{ 50)}$   
 $-uconverse \quad :: logic \Rightarrow logic \text{ ((~) [1000] 999)}$   
 $-ucarrier \quad :: type \Rightarrow logic \text{ ([-]_T)}$   
 $-uid \quad :: type \Rightarrow logic \text{ (id[-])}$   
 $-uproduct \quad :: logic \Rightarrow logic \Rightarrow logic \text{ (infixr } \times_u \text{ 80)}$   
 $-urelcomp \quad :: logic \Rightarrow logic \Rightarrow logic \text{ (infixr } ;_u \text{ 75)}$

#### translations

$finite_u(x) == CONST \text{ uop } (CONST \text{ finite}) x$   
 $\{\}_u == \ll \{\} \gg$   
 $insert_u x xs == CONST \text{ bop } CONST \text{ insert } x xs$   
 $\{x, xs\}_u == insert_u x \{xs\}_u$   
 $\{x\}_u == insert_u x \ll \{\} \gg$   
 $A \cup_u B == CONST \text{ bop } (\cup) A B$   
 $A \cap_u B == CONST \text{ bop } (\cap) A B$

$f(\downarrow A)_u == \text{CONST bop } \text{CONST image } f \ A$   
 $A \subset_u B == \text{CONST bop } (\subset) \ A \ B$   
 $f \subset_u g <= \text{CONST bop } (\subset_p) \ f \ g$   
 $f \subset_u g <= \text{CONST bop } (\subset_f) \ f \ g$   
 $A \subseteq_u B == \text{CONST bop } (\subseteq) \ A \ B$   
 $f \subseteq_u g <= \text{CONST bop } (\subseteq_p) \ f \ g$   
 $f \subseteq_u g <= \text{CONST bop } (\subseteq_f) \ f \ g$   
 $P^{\sim}_u == \text{CONST uop } \text{CONST converse } P$   
 $[a]_T == \ll \text{CONST set-of } \text{TYPE}('a) \gg$   
 $\text{id}[a] == \ll \text{CONST Id-on } (\text{CONST set-of } \text{TYPE}('a)) \gg$   
 $A \times_u B == \text{CONST bop } \text{CONST Product-Type.Times } A \ B$   
 $A ;_u B == \text{CONST bop } \text{CONST relcomp } A \ B$

**syntax** — Partial functions

$\text{-umap-plus} :: \text{logic} \Rightarrow \text{logic} \Rightarrow \text{logic} \ (\text{infixl } \oplus_u \ 85)$   
 $\text{-umap-minus} :: \text{logic} \Rightarrow \text{logic} \Rightarrow \text{logic} \ (\text{infixl } \ominus_u \ 85)$

**translations**

$f \oplus_u g \Rightarrow (f :: ((-, -) \text{ pfun}, -) \text{ uexpr}) + g$   
 $f \ominus_u g \Rightarrow (f :: ((-, -) \text{ pfun}, -) \text{ uexpr}) - g$

**syntax** — Sum types

$\text{-uinl} :: \text{logic} \Rightarrow \text{logic} \ (\text{inl}_u '(-))$   
 $\text{-uinr} :: \text{logic} \Rightarrow \text{logic} \ (\text{inr}_u '(-))$

**translations**

$\text{inl}_u(x) == \text{CONST uop } \text{CONST Inl } x$   
 $\text{inr}_u(x) == \text{CONST uop } \text{CONST Inr } x$

## 4.2 Lifting set collectors

We provide syntax for various types of set collectors, including intervals and the Z-style set comprehension which is purpose built as a new lifted definition.

**syntax**

$\text{-uset-atLeastAtMost} :: ('a, 'α) \text{ uexpr} \Rightarrow ('a, 'α) \text{ uexpr} \Rightarrow ('a \text{ set}, 'α) \text{ uexpr} \ ((1\{-..\}-\}_u))$   
 $\text{-uset-atLeastLessThan} :: ('a, 'α) \text{ uexpr} \Rightarrow ('a, 'α) \text{ uexpr} \Rightarrow ('a \text{ set}, 'α) \text{ uexpr} \ ((1\{-..<\}_u))$   
 $\text{-uset-compr} :: \text{pttrn} \Rightarrow ('a \text{ set}, 'α) \text{ uexpr} \Rightarrow (\text{bool}, 'α) \text{ uexpr} \Rightarrow ('b, 'α) \text{ uexpr} \Rightarrow ('b \text{ set}, 'α) \text{ uexpr} \ ((1\{- :/ - \mid - \cdot / -\}_u))$   
 $\text{-uset-compr-nset} :: \text{pttrn} \Rightarrow (\text{bool}, 'α) \text{ uexpr} \Rightarrow ('b, 'α) \text{ uexpr} \Rightarrow ('b \text{ set}, 'α) \text{ uexpr} \ ((1\{- \mid - \cdot / -\}_u))$

**lift-definition**  $\text{ZedSetCompr} ::$

$(('a \text{ set}, 'α) \text{ uexpr} \Rightarrow ('a \Rightarrow (\text{bool}, 'α) \text{ uexpr} \times ('b, 'α) \text{ uexpr}) \Rightarrow ('b \text{ set}, 'α) \text{ uexpr})$   
 $\text{is } \lambda A \ PF \ b. \{ \text{snd } (PF \ x) \ b \mid x. x \in A \ b \wedge \text{fst } (PF \ x) \ b \} .$

**translations**

$\{x..y\}_u == \text{CONST bop } \text{CONST atLeastAtMost } x \ y$   
 $\{x..<y\}_u == \text{CONST bop } \text{CONST atLeastLessThan } x \ y$   
 $\{x \mid P \cdot F\}_u == \text{CONST ZedSetCompr } (\text{CONST lit } \text{CONST UNIV}) \ (\lambda x. (P, F))$   
 $\{x : A \mid P \cdot F\}_u == \text{CONST ZedSetCompr } A \ (\lambda x. (P, F))$

## 4.3 Lifting limits

We also lift the following functions on topological spaces for taking function limits, and describing continuity.

**definition** *ulim-left* :: 'a::order-topology  $\Rightarrow$  ('a  $\Rightarrow$  'b)  $\Rightarrow$  'b::t2-space **where**  
[uepr-defs]: *ulim-left* = ( $\lambda p f. \text{Lim } (\text{at-left } p) f$ )

**definition** *ulim-right* :: 'a::order-topology  $\Rightarrow$  ('a  $\Rightarrow$  'b)  $\Rightarrow$  'b::t2-space **where**  
[uepr-defs]: *ulim-right* = ( $\lambda p f. \text{Lim } (\text{at-right } p) f$ )

**definition** *ucont-on* :: ('a::topological-space  $\Rightarrow$  'b::topological-space)  $\Rightarrow$  'a set  $\Rightarrow$  bool **where**  
[uepr-defs]: *ucont-on* = ( $\lambda f A. \text{continuous-on } A f$ )

**syntax**

-*ulim-left* :: id  $\Rightarrow$  logic  $\Rightarrow$  logic  $\Rightarrow$  logic (*lim<sub>u</sub>*'(-  $\rightarrow$  -<sup>-</sup>)'(-'))  
-*ulim-right* :: id  $\Rightarrow$  logic  $\Rightarrow$  logic  $\Rightarrow$  logic (*lim<sub>u</sub>*'(-  $\rightarrow$  -<sup>+</sup>)'(-'))  
-*ucont-on* :: logic  $\Rightarrow$  logic  $\Rightarrow$  logic (**infix** *cont-on<sub>u</sub>* 90)

**translations**

*lim<sub>u</sub>*( $x \rightarrow p^-$ )( $e$ ) == *CONST bop CONST ulim-left p* ( $\lambda x \cdot e$ )  
*lim<sub>u</sub>*( $x \rightarrow p^+$ )( $e$ ) == *CONST bop CONST ulim-right p* ( $\lambda x \cdot e$ )  
*f cont-on<sub>u</sub> A* == *CONST bop CONST continuous-on A f*

**lemma** *uset-minus-empty* [simp]:  $x - \{\}_u = x$   
**by** (simp add: uepr-defs, transfer, simp)

**lemma** *uinter-empty-1* [simp]:  $x \cap_u \{\}_u = \{\}_u$   
**by** (transfer, simp)

**lemma** *uinter-empty-2* [simp]:  $\{\}_u \cap_u x = \{\}_u$   
**by** (transfer, simp)

**lemma** *union-empty-1* [simp]:  $\{\}_u \cup_u x = x$   
**by** (transfer, simp)

**lemma** *union-insert* [simp]:  $(\text{bop insert } x A) \cup_u B = \text{bop insert } x (A \cup_u B)$   
**by** (transfer, simp)

**lemma** *ulist-filter-empty* [simp]:  $x \upharpoonright_u \{\}_u = \langle \rangle$   
**by** (transfer, simp)

**lemma** *tail-cons* [simp]:  $\text{tail}_u(\langle x \rangle \hat{\ }_u xs) = xs$   
**by** (transfer, simp)

**lemma** *uconcat-units* [simp]:  $\langle \rangle \hat{\ }_u xs = xs \text{ xs } \hat{\ }_u \langle \rangle = xs$   
**by** (transfer, simp)+

**end**

## 5 Unrestriction

**theory** *utp-unrest*  
**imports** *utp-expr-insts*  
**begin**

### 5.1 Definitions and Core Syntax

Unrestriction is an encoding of semantic freshness that allows us to reason about the presence of variables in predicates without being concerned with abstract syntax trees. An expression  $p$

is unrestricted by lens  $x$ , written  $x \# p$ , if altering the value of  $x$  has no effect on the valuation of  $p$ . This is a sufficient notion to prove many laws that would ordinarily rely on an  $fv$  function.

Unrestriction was first defined in the work of Marcel Oliveira [27, 26] in his UTP mechanisation in *ProofPowerZ*. Our definition modifies his in that our variables are semantically characterised as lenses, and supported by the lens laws, rather than named syntactic entities. We effectively fuse the ideas from both Feliachi [9] and Oliveira's [26] mechanisations of the UTP, the former being also purely semantic in nature.

We first set up overloaded syntax for unrestricted, as several concepts will have this defined.

**consts**

$unrest :: 'a \Rightarrow 'b \Rightarrow bool$

**syntax**

$-unrest :: salpha \Rightarrow logic \Rightarrow logic \Rightarrow logic$  (**infix**  $\#$  20)

**translations**

$-unrest\ x\ p == CONST\ unrest\ x\ p$

$-unrest\ (-salphaset\ (-salphamk\ (x +_L\ y)))\ P <= -unrest\ (x +_L\ y)\ P$

Our syntax translations support both variables and variable sets such that we can write down predicates like  $\&x \# P$  and also  $\{\&x, \&y, \&z\} \# P$ .

We set up a simple tactic for discharging unrestricted conjectures using a simplification set.

**named-theorems**  $unrest$

**method**  $unrest-tac = (simp\ add: unrest)?$

Unrestriction for expressions is defined as a lifted construct using the underlying lens operations. It states that lens  $x$  is unrestricted by expression  $e$  provided that, for any state-space binding  $b$  and variable valuation  $v$ , the value which the expression evaluates to is unaltered if we set  $x$  to  $v$  in  $b$ . In other words, we cannot effect the behaviour of  $e$  by changing  $x$ . Thus  $e$  does not observe the portion of state-space characterised by  $x$ . We add this definition to our overloaded constant.

**lift-definition**  $unrest-uepr :: ('a \Longrightarrow 'a) \Rightarrow ('b, 'a) uepr \Rightarrow bool$

**is**  $\lambda x\ e. \forall\ b\ v. e\ (put_x\ b\ v) = e\ b$ .

**ad hoc-overloading**

$unrest\ unrest-uepr$

**lemma**  $unrest-expr-alt-def$ :

$weak-lens\ x \Longrightarrow (x \# P) = (\forall\ b\ b'. \llbracket P \rrbracket_e (b \oplus_L b' \text{ on } x) = \llbracket P \rrbracket_e b)$

**by** (*transfer, metis lens-override-def weak-lens.put-get*)

## 5.2 Unrestriction laws

We now prove unrestricted laws for the key constructs of our expression model. Many of these depend on lens properties and so variously employ the assumptions *mwb-lens* and *vwb-lens*, depending on the number of assumptions from the lenses theory is required.

Firstly, we prove a general property – if  $x$  and  $y$  are both unrestricted in  $P$ , then their composition is also unrestricted in  $P$ . One can interpret the composition here as a union – if the two sets of variables  $x$  and  $y$  are unrestricted, then so is their union.

**lemma**  $unrest-var-comp\ [unrest]$ :

$\llbracket x \# P; y \# P \rrbracket \Longrightarrow x;y \# P$

**by** (*transfer, simp add: lens-defs*)

**lemma** *unrest-svar* [*unrest*]:  $(\&x \# P) \longleftrightarrow (x \# P)$   
**by** (*transfer*, *simp add: lens-defs*)

No lens is restricted by a literal, since it returns the same value for any state binding.

**lemma** *unrest-lit* [*unrest*]:  $x \# \ll v \gg$   
**by** (*transfer*, *simp*)

If one lens is smaller than another, then any unrestriction on the larger lens implies unrestriction on the smaller.

**lemma** *unrest-sublens*:  
**fixes**  $P :: ('a, 'α) uexpr$   
**assumes**  $x \# P \ y \subseteq_L x$   
**shows**  $y \# P$   
**using** *assms*  
**by** (*transfer*, *metis (no-types, lifting) lens.select-convs(2) lens-comp-def sublens-def*)

If two lenses are equivalent, and thus they characterise the same state-space regions, then clearly unrestrictions over them are equivalent.

**lemma** *unrest-equiv*:  
**fixes**  $P :: ('a, 'α) uexpr$   
**assumes**  $mwb\text{-}lens \ y \ x \approx_L y \ x \# P$   
**shows**  $y \# P$   
**by** (*metis assms lens-equiv-def sublens-pres-mwb sublens-put-put unrest-uexpr.rep-eq*)

If we can show that an expression is unrestricted on a bijective lens, then is unrestricted on the entire state-space.

**lemma** *bij-lens-unrest-all*:  
**fixes**  $P :: ('a, 'α) uexpr$   
**assumes**  $bij\text{-}lens \ X \ X \# P$   
**shows**  $\Sigma \# P$   
**using** *assms bij-lens-equiv-id lens-equiv-def unrest-sublens* **by** *blast*

**lemma** *bij-lens-unrest-all-eq*:  
**fixes**  $P :: ('a, 'α) uexpr$   
**assumes**  $bij\text{-}lens \ X$   
**shows**  $(\Sigma \# P) \longleftrightarrow (X \# P)$   
**by** (*meson assms bij-lens-equiv-id lens-equiv-def unrest-sublens*)

If an expression is unrestricted by all variables, then it is unrestricted by any variable

**lemma** *unrest-all-var*:  
**fixes**  $e :: ('a, 'α) uexpr$   
**assumes**  $\Sigma \# e$   
**shows**  $x \# e$   
**by** (*metis assms id-lens-def lens.simps(2) unrest-uexpr.rep-eq*)

We can split an unrestriction composed by lens plus

**lemma** *unrest-plus-split*:  
**fixes**  $P :: ('a, 'α) uexpr$   
**assumes**  $x \bowtie y \ vwb\text{-}lens \ x \ vwb\text{-}lens \ y$   
**shows**  $unrest \ (x +_L y) \ P \longleftrightarrow (x \# P) \wedge (y \# P)$   
**using** *assms*  
**by** (*meson lens-plus-right-sublens lens-plus-ub sublens-refl unrest-sublens unrest-var-comp vwb-lens-wb*)

The following laws demonstrate the primary motivation for lens independence: a variable expression is unrestricted by another variable only when the two variables are independent. Lens independence thus effectively allows us to semantically characterise when two variables, or sets of variables, are different.

**lemma** *unrest-var* [*unrest*]:  $\llbracket \text{mwb-lens } x; x \bowtie y \rrbracket \Longrightarrow y \# \text{var } x$   
**by** (*transfer*, *auto*)

**lemma** *unrest-iuvar* [*unrest*]:  $\llbracket \text{mwb-lens } x; x \bowtie y \rrbracket \Longrightarrow \$y \# \$x$   
**by** (*simp add: unrest-var*)

**lemma** *unrest-ouvar* [*unrest*]:  $\llbracket \text{mwb-lens } x; x \bowtie y \rrbracket \Longrightarrow \$y' \# \$x'$   
**by** (*simp add: unrest-var*)

The following laws follow automatically from independence of input and output variables.

**lemma** *unrest-iuvar-ouvar* [*unrest*]:  
**fixes**  $x :: ('a \Longrightarrow 'a)$   
**assumes** *mwb-lens*  $y$   
**shows**  $\$x \# \$y'$   
**by** (*metis prod.collapse unrest-uexpr.rep-eq var.rep-eq var-lookup-out var-update-in*)

**lemma** *unrest-ouvar-iuvar* [*unrest*]:  
**fixes**  $x :: ('a \Longrightarrow 'a)$   
**assumes** *mwb-lens*  $y$   
**shows**  $\$x' \# \$y$   
**by** (*metis prod.collapse unrest-uexpr.rep-eq var.rep-eq var-lookup-in var-update-out*)

Unrestriction distributes through the various function lifting expression constructs; this allows us to prove unrestrictions for the majority of the expression language.

**lemma** *unrest-uop* [*unrest*]:  $x \# e \Longrightarrow x \# \text{uop } f e$   
**by** (*transfer*, *simp*)

**lemma** *unrest-bop* [*unrest*]:  $\llbracket x \# u; x \# v \rrbracket \Longrightarrow x \# \text{bop } f u v$   
**by** (*transfer*, *simp*)

**lemma** *unrest-trop* [*unrest*]:  $\llbracket x \# u; x \# v; x \# w \rrbracket \Longrightarrow x \# \text{trop } f u v w$   
**by** (*transfer*, *simp*)

**lemma** *unrest-qtop* [*unrest*]:  $\llbracket x \# u; x \# v; x \# w; x \# y \rrbracket \Longrightarrow x \# \text{qtop } f u v w y$   
**by** (*transfer*, *simp*)

For convenience, we also prove unrestriction rules for the bespoke operators on equality, numbers, arithmetic etc.

**lemma** *unrest-eq* [*unrest*]:  $\llbracket x \# u; x \# v \rrbracket \Longrightarrow x \# u =_u v$   
**by** (*simp add: eq-upred-def*, *transfer*, *simp*)

**lemma** *unrest-zero* [*unrest*]:  $x \# 0$   
**by** (*simp add: unrest-lit zero-uexpr-def*)

**lemma** *unrest-one* [*unrest*]:  $x \# 1$   
**by** (*simp add: one-uexpr-def unrest-lit*)

**lemma** *unrest-numeral* [*unrest*]:  $x \# (\text{numeral } n)$   
**by** (*simp add: numeral-uexpr-simp unrest-lit*)

**lemma** *unrest-sgn* [*unrest*]:  $x \# u \implies x \# \text{sgn } u$   
 by (*simp add: sgn-ueexpr-def unrest-uop*)

**lemma** *unrest-abs* [*unrest*]:  $x \# u \implies x \# \text{abs } u$   
 by (*simp add: abs-ueexpr-def unrest-uop*)

**lemma** *unrest-plus* [*unrest*]:  $\llbracket x \# u; x \# v \rrbracket \implies x \# u + v$   
 by (*simp add: plus-ueexpr-def unrest*)

**lemma** *unrest-uminus* [*unrest*]:  $x \# u \implies x \# -u$   
 by (*simp add: uminus-ueexpr-def unrest*)

**lemma** *unrest-minus* [*unrest*]:  $\llbracket x \# u; x \# v \rrbracket \implies x \# u - v$   
 by (*simp add: minus-ueexpr-def unrest*)

**lemma** *unrest-times* [*unrest*]:  $\llbracket x \# u; x \# v \rrbracket \implies x \# u * v$   
 by (*simp add: times-ueexpr-def unrest*)

**lemma** *unrest-divide* [*unrest*]:  $\llbracket x \# u; x \# v \rrbracket \implies x \# u / v$   
 by (*simp add: divide-ueexpr-def unrest*)

**lemma** *unrest-case-prod* [*unrest*]:  $\llbracket \bigwedge i j. x \# P i j \rrbracket \implies x \# \text{case-prod } P v$   
 by (*simp add: prod.split-sel-asm*)

For a  $\lambda$ -term we need to show that the characteristic function expression does not restrict  $v$  for any input value  $x$ .

**lemma** *unrest-ulambda* [*unrest*]:  
 $\llbracket \bigwedge x. v \# F x \rrbracket \implies v \# (\lambda x. F x)$   
 by (*transfer, simp*)

end

## 6 Used-by

**theory** *utp-usedby*  
**imports** *utp-unrest*  
**begin**

The used-by predicate is the dual of unrestriction. It states that the given lens is an upper-bound on the size of state space the given expression depends on. It is similar to stating that the lens is a valid alphabet for the predicate. For convenience, and because the predicate uses a similar form, we will reuse much of unrestriction's infrastructure.

**consts**  
*usedBy* ::  $'a \Rightarrow 'b \Rightarrow \text{bool}$

**syntax**  
 $\text{-usedBy} :: \text{salpha} \Rightarrow \text{logic} \Rightarrow \text{logic} \Rightarrow \text{logic} \text{ (infix } \bowtie 20)$

**translations**  
 $\text{-usedBy } x \text{ } p == \text{CONST usedBy } x \text{ } p$   
 $\text{-usedBy } (-\text{salphaset } (-\text{salphamk } (x +_L y))) \text{ } P <= \text{-usedBy } (x +_L y) \text{ } P$

**lift-definition** *usedBy-ueexpr* ::  $('b \implies 'a) \Rightarrow ('a, 'a) \text{ ueexpr} \Rightarrow \text{bool}$   
**is**  $\lambda x e. (\forall b b'. e (b' \oplus_L b \text{ on } x) = e b) .$

**ad hoc-overloading** *usedBy usedBy-uepr*

**lemma** *usedBy-lit* [unrest]:  $x \Vdash \llbracket v \rrbracket$   
 by (*transfer*, *simp*)

**lemma** *usedBy-sublens*:  
 fixes  $P :: ('a, 'α) \text{uepr}$   
 assumes  $x \Vdash P \ x \subseteq_L y \text{ vwb-lens } y$   
 shows  $y \Vdash P$   
 using *assms*  
 by (*transfer*, *auto*, *metis* *Lens-Order.lens-override-idem lens-override-def sublens-obs-get vwb-lens-mwb*)

**lemma** *usedBy-svar* [unrest]:  $x \Vdash P \implies \&x \Vdash P$   
 by (*transfer*, *simp* *add: lens-defs*)

**lemma** *usedBy-lens-plus-1* [unrest]:  $x \Vdash P \implies x; y \Vdash P$   
 by (*transfer*, *simp* *add: lens-defs*)

**lemma** *usedBy-lens-plus-2* [unrest]:  $\llbracket x \bowtie y; y \Vdash P \rrbracket \implies x; y \Vdash P$   
 by (*transfer*, *auto* *simp* *add: lens-defs lens-indep-comm*)

Linking used-by to unrestriction: if  $x$  is used-by  $P$ , and  $x$  is independent of  $y$ , then  $P$  cannot depend on any variable in  $y$ .

**lemma** *usedBy-indep-uses*:  
 fixes  $P :: ('a, 'α) \text{uepr}$   
 assumes  $x \Vdash P \ x \bowtie y$   
 shows  $y \nVdash P$   
 using *assms* by (*transfer*, *auto*, *metis* *lens-indep-get lens-override-def*)

**lemma** *usedBy-var* [unrest]:  
 assumes  $\text{vwb-lens } x \ y \subseteq_L x$   
 shows  $x \Vdash \text{var } y$   
 using *assms*  
 by (*transfer*, *simp* *add: uepr-defs pr-var-def*)  
 (*metis* *lens-override-def sublens-obs-get vwb-lens-def wb-lens.get-put*)

**lemma** *usedBy-uop* [unrest]:  $x \Vdash e \implies x \Vdash \text{uop } f \ e$   
 by (*transfer*, *simp*)

**lemma** *usedBy-bop* [unrest]:  $\llbracket x \Vdash u; x \Vdash v \rrbracket \implies x \Vdash \text{bop } f \ u \ v$   
 by (*transfer*, *simp*)

**lemma** *usedBy-trop* [unrest]:  $\llbracket x \Vdash u; x \Vdash v; x \Vdash w \rrbracket \implies x \Vdash \text{trop } f \ u \ v \ w$   
 by (*transfer*, *simp*)

**lemma** *usedBy-qtop* [unrest]:  $\llbracket x \Vdash u; x \Vdash v; x \Vdash w; x \Vdash y \rrbracket \implies x \Vdash \text{qtop } f \ u \ v \ w \ y$   
 by (*transfer*, *simp*)

For convenience, we also prove used-by rules for the bespoke operators on equality, numbers, arithmetic etc.

**lemma** *usedBy-eq* [unrest]:  $\llbracket x \Vdash u; x \Vdash v \rrbracket \implies x \Vdash u =_u v$   
 by (*simp* *add: eq-upred-def*, *transfer*, *simp*)

**lemma** *usedBy-zero* [unrest]:  $x \Vdash 0$



```

by (simp add: usedBy-lit zero-uepr-def)

lemma usedBy-one [unrest]:  $x \Vdash 1$ 
  by (simp add: one-uepr-def usedBy-lit)

lemma usedBy-numeral [unrest]:  $x \Vdash (\text{numeral } n)$ 
  by (simp add: numeral-uepr-simp usedBy-lit)

lemma usedBy-sgn [unrest]:  $x \Vdash u \implies x \Vdash \text{sgn } u$ 
  by (simp add: sgn-uepr-def usedBy-uop)

lemma usedBy-abs [unrest]:  $x \Vdash u \implies x \Vdash \text{abs } u$ 
  by (simp add: abs-uepr-def usedBy-uop)

lemma usedBy-plus [unrest]:  $\llbracket x \Vdash u; x \Vdash v \rrbracket \implies x \Vdash u + v$ 
  by (simp add: plus-uepr-def unrest)

lemma usedBy-uminus [unrest]:  $x \Vdash u \implies x \Vdash - u$ 
  by (simp add: uminus-uepr-def unrest)

lemma usedBy-minus [unrest]:  $\llbracket x \Vdash u; x \Vdash v \rrbracket \implies x \Vdash u - v$ 
  by (simp add: minus-uepr-def unrest)

lemma usedBy-times [unrest]:  $\llbracket x \Vdash u; x \Vdash v \rrbracket \implies x \Vdash u * v$ 
  by (simp add: times-uepr-def unrest)

lemma usedBy-divide [unrest]:  $\llbracket x \Vdash u; x \Vdash v \rrbracket \implies x \Vdash u / v$ 
  by (simp add: divide-uepr-def unrest)

lemma usedBy-ulambda [unrest]:
   $\llbracket \bigwedge x. v \Vdash F x \rrbracket \implies v \Vdash (\lambda x. F x)$ 
  by (transfer, simp)

lemma unrest-var-sep [unrest]:
   $\text{vwb-lens } x \implies x \Vdash \&x:y$ 
  by (transfer, simp add: lens-defs)

end

```

## 7 Substitution

```

theory utp-subst
imports
  utp-expr
  utp-unrest
begin

```

### 7.1 Substitution definitions

Variable substitution, like unrestriction, will be characterised semantically using lenses and state-spaces. Effectively a substitution  $\sigma$  is simply a function on the state-space which can be applied to an expression  $e$  using the syntax  $\sigma \dagger e$ . We introduce a polymorphic constant that will be used to represent application of a substitution, and also a set of theorems to represent laws.

**consts**

$usubst :: 's \Rightarrow 'a \Rightarrow 'b \text{ (infixr } \dagger \ 80)$

**named-theorems**  $usubst$

A substitution is simply a transformation on the alphabet; it shows how variables should be mapped to different values. Most of the time these will be homogeneous functions but for flexibility we also allow some operations to be heterogeneous.

**type-synonym**  $('a, 'b) psubst = 'a \Rightarrow 'b$

**type-synonym**  $'a usubst = 'a \Rightarrow 'a$

Application of a substitution simply applies the function  $\sigma$  to the state binding  $b$  before it is handed to  $e$  as an input. This effectively ensures all variables are updated in  $e$ .

**lift-definition**  $subst :: ('a, 'b) psubst \Rightarrow ('a, 'b) uexpr \Rightarrow ('a, 'a) uexpr \text{ is}$   
 $\lambda \sigma e b. e (\sigma b) .$

**ad hoc-overloading**

$usubst \ subst$

Substitutions can be updated by associating variables with expressions. We thus create an additional polymorphic constant to represent updating the value of a variable to an expression in a substitution, where the variable is modelled by type  $'v$ . This again allows us to support different notions of variables, such as deep variables, later.

**consts**  $subst-upd :: ('a, 'b) psubst \Rightarrow 'v \Rightarrow ('a, 'a) uexpr \Rightarrow ('a, 'b) psubst$

The following function takes a substitution from state-space  $'\alpha$  to  $'\beta$ , a lens with source  $'\beta$  and view  $''a$ , and an expression over  $'\alpha$  and returning a value of type  $''a$ , and produces an updated substitution. It does this by constructing a substitution function that takes state binding  $b$ , and updates the state first by applying the original substitution  $\sigma$ , and then updating the part of the state associated with lens  $x$  with expression evaluated in the context of  $b$ . This effectively means that  $x$  is now associated with expression  $v$ . We add this definition to our overloaded constant.

**definition**  $subst-upd-uvar :: ('a, 'b) psubst \Rightarrow ('a \Rightarrow 'b) \Rightarrow ('a, 'a) uexpr \Rightarrow ('a, 'b) psubst \text{ where}$   
 $subst-upd-uvar \ \sigma \ x \ v = (\lambda b. put_x (\sigma b) (\llbracket v \rrbracket_e b))$

**ad hoc-overloading**

$subst-upd \ subst-upd-uvar$

The next function looks up the expression associated with a variable in a substitution by use of the *get* lens function.

**lift-definition**  $usubst-lookup :: ('a, 'b) psubst \Rightarrow ('a \Rightarrow 'b) \Rightarrow ('a, 'a) uexpr \Rightarrow (\langle - \rangle_s)$   
**is**  $\lambda \sigma x b. get_x (\sigma b) .$

Substitutions also exhibit a natural notion of unrestriction which states that  $\sigma$  does not restrict  $x$  if application of  $\sigma$  to an arbitrary state  $\rho$  will not effect the valuation of  $x$ . Put another way, it requires that *put* and the substitution commute.

**definition**  $unrest-usubst :: ('a \Rightarrow 'a) \Rightarrow 'a usubst \Rightarrow bool$   
**where**  $unrest-usubst \ x \ \sigma = (\forall \ \varrho \ v. \ \sigma (put_x \ \varrho \ v) = put_x (\sigma \ \varrho) \ v)$

**ad hoc-overloading**

$unrest \ unrest-usubst$

A conditional substitution deterministically picks one of the two substitutions based on a Boolean expression which is evaluated on the present state-space. It is analogous to a functional if-then-else.

**definition**  $\text{cond-subst} :: 'a \text{ usubst} \Rightarrow (\text{bool}, 'a) \text{ uexpr} \Rightarrow 'a \text{ usubst} \Rightarrow 'a \text{ usubst} ((\mathcal{B} \cdot \triangleleft \cdot \triangleright_s / -) [52, 0, 53] 52)$  **where**  
 $\text{cond-subst } \sigma \ b \ \varrho = (\lambda s. \text{ if } \llbracket b \rrbracket_e s \text{ then } \sigma(s) \text{ else } \varrho(s))$

Parallel substitutions allow us to divide the state space into three segments using two lens, A and B. They correspond to the part of the state that should be updated by the respective substitution. The two lenses should be independent. If any part of the state is not covered by either lenses then this area is left unchanged (framed).

**definition**  $\text{par-subst} :: 'a \text{ usubst} \Rightarrow ('a \Rightarrow 'a) \Rightarrow ('b \Rightarrow 'a) \Rightarrow 'a \text{ usubst} \Rightarrow 'a \text{ usubst}$  **where**  
 $\text{par-subst } \sigma_1 \ A \ B \ \sigma_2 = (\lambda s. (s \oplus_L (\sigma_1 \ s) \text{ on } A) \oplus_L (\sigma_2 \ s) \text{ on } B)$

## 7.2 Syntax translations

We support two kinds of syntax for substitutions, one where we construct a substitution using a maplet-style syntax, with variables mapping to expressions. Such a constructed substitution can be applied to an expression. Alternatively, we support the more traditional notation,  $P\llbracket v/x \rrbracket$ , which also support multiple simultaneous substitutions. We have to use double square brackets as the single ones are already well used.

We set up non-terminals to represent a single substitution maplet, a sequence of maplets, a list of expressions, and a list of alphabets. The parser effectively uses *subst-upd* to construct substitutions from multiple variables.

**nonterminal** *maplet and maplets and uexp and uexprs and salphas*

**syntax**

```
-maplet  :: [salpha, 'a] => maplet      (- /↦s/ -)
           :: maplet => maplets         (-)
-SMaplets :: [maplet, maplets] => maplets (-, / -)
-SubstUpd :: ['m usubst, maplets] => 'm usubst (-/'(-) [900,0] 900)
-Subst    :: maplets => 'a ↦ 'b        ((1[-]))
-psubst   :: [logic, svar, uexprs] => logic
-subst    :: logic => uexprs => salphas => logic (([-]'(-)] [990,0,0] 991)
-uexp-l   :: logic => uexp (- [64] 64)
-uexprs   :: [uexp, uexprs] => uexprs (-, / -)
           :: uexp => uexprs (-)
-salphas  :: [salpha, salphas] => salphas (-, / -)
           :: salpha => salphas (-)
-par-subst :: logic => salpha => salpha => logic => logic (- [-]_s - [100,0,0,101] 101)
```

**translations**

```
-SubstUpd m (-SMaplets xy ms) == -SubstUpd (-SubstUpd m xy) ms
-SubstUpd m (-maplet x y)    == CONST subst-upd m x y
-Subst ms                    == -SubstUpd (CONST id) ms
-Subst (-SMaplets ms1 ms2)   <= -SubstUpd (-Subst ms1) ms2
-SMaplets ms1 (-SMaplets ms2 ms3) <= -SMaplets (-SMaplets ms1 ms2) ms3
-subst P es vs => CONST subst (-psubst (CONST id) vs es) P
-psubst m (-salphas x xs) (-uexprs v vs) => -psubst (-psubst m x v) xs vs
-psubst m x v => CONST subst-upd m x v
-subst P v x <= CONST usubst (CONST subst-upd (CONST id) x v) P
-subst P v x <= -subst P (-spvar x) v
-par-subst σ1 A B σ2 == CONST par-subst σ1 A B σ2
```

-uexp-l e => e

Thus we can write things like  $\sigma(x \mapsto_s v)$  to update a variable  $x$  in  $\sigma$  with expression  $v$ ,  $[x \mapsto_s e, y \mapsto_s f]$  to construct a substitution with two variables, and finally  $P[v/x]$ , the traditional syntax.

We can now express deletion of a substitution maplet.

**definition** *subst-del* ::  $'\alpha \text{ usubst} \Rightarrow ('a \Longrightarrow '\alpha) \Rightarrow '\alpha \text{ usubst}$  (**infix**  $-_s$  85) **where**  
*subst-del*  $\sigma \ x = \sigma(x \mapsto_s \&x)$

### 7.3 Substitution Application Laws

We set up a simple substitution tactic that applies substitution and unrestriction laws

**method** *subst-tac* = (*simp add: usubst unrest*)?

Evaluation of a substitution expression involves application of the substitution to different variables. Thus we first prove laws for these cases. The simplest substitution, *id*, when applied to any variable  $x$  simply returns the variable expression, since *id* has no effect.

**lemma** *usubst-lookup-id* [*usubst*]:  $\langle id \rangle_s x = \text{var } x$   
**by** (*transfer, simp*)

**lemma** *subst-upd-id-lam* [*usubst*]: *subst-upd*  $(\lambda x. x) x v = \text{subst-upd } id x v$   
**by** (*simp add: id-def*)

A substitution update naturally yields the given expression.

**lemma** *usubst-lookup-upd* [*usubst*]:  
**assumes** *weak-lens x*  
**shows**  $\langle \sigma(x \mapsto_s v) \rangle_s x = v$   
**using** *assms*  
**by** (*simp add: subst-upd-uvar-def, transfer*) (*simp*)

**lemma** *usubst-lookup-upd-pr-var* [*usubst*]:  
**assumes** *weak-lens x*  
**shows**  $\langle \sigma(x \mapsto_s v) \rangle_s (\text{pr-var } x) = v$   
**using** *assms*  
**by** (*simp add: subst-upd-uvar-def pr-var-def, transfer*) (*simp*)

Substitution update is idempotent.

**lemma** *usubst-upd-idem* [*usubst*]:  
**assumes** *mwb-lens x*  
**shows**  $\sigma(x \mapsto_s u, x \mapsto_s v) = \sigma(x \mapsto_s v)$   
**by** (*simp add: subst-upd-uvar-def assms comp-def*)

**lemma** *usubst-upd-idem-sub* [*usubst*]:  
**assumes**  $x \subseteq_L y$  *mwb-lens y*  
**shows**  $\sigma(x \mapsto_s u, y \mapsto_s v) = \sigma(y \mapsto_s v)$   
**by** (*simp add: subst-upd-uvar-def assms comp-def fun-eq-iff sublens-put-put*)

Substitution updates commute when the lenses are independent.

**lemma** *usubst-upd-comm*:  
**assumes**  $x \bowtie y$   
**shows**  $\sigma(x \mapsto_s u, y \mapsto_s v) = \sigma(y \mapsto_s v, x \mapsto_s u)$   
**using** *assms*  
**by** (*rule-tac ext, auto simp add: subst-upd-uvar-def assms comp-def lens-indep-comm*)

**lemma** *usubst-upd-comm2*:  
**assumes**  $z \bowtie y$   
**shows**  $\sigma(x \mapsto_s u, y \mapsto_s v, z \mapsto_s s) = \sigma(x \mapsto_s u, z \mapsto_s s, y \mapsto_s v)$   
**using** *assms*  
**by** (*rule-tac ext*, *auto simp add: subst-upd-uvar-def assms comp-def lens-indep-comm*)

**lemma** *subst-upd-pr-var*:  $s(\&x \mapsto_s v) = s(x \mapsto_s v)$   
**by** (*simp add: pr-var-def*)

A substitution which swaps two independent variables is an injective function.

**lemma** *swap-usubst-inj*:  
**fixes**  $x\ y :: ('a \Rightarrow 'a)$   
**assumes** *vwb-lens*  $x\ \text{vwb-lens}\ y\ x \bowtie y$   
**shows** *inj*  $[x \mapsto_s \&y, y \mapsto_s \&x]$   
**proof** (*rule injI*)  
**fix**  $b_1 :: 'a$  **and**  $b_2 :: 'a$   
**assume**  $[x \mapsto_s \&y, y \mapsto_s \&x]\ b_1 = [x \mapsto_s \&y, y \mapsto_s \&x]\ b_2$   
**hence**  $a: \text{put}_y (\text{put}_x b_1 (\llbracket \&y \rrbracket_e b_1)) (\llbracket \&x \rrbracket_e b_1) = \text{put}_y (\text{put}_x b_2 (\llbracket \&y \rrbracket_e b_2)) (\llbracket \&x \rrbracket_e b_2)$   
**by** (*auto simp add: subst-upd-uvar-def*)  
**then have**  $(\forall a\ b\ c. \text{put}_x (\text{put}_y a\ b)\ c = \text{put}_y (\text{put}_x a\ c)\ b) \wedge$   
 $(\forall a\ b. \text{get}_x (\text{put}_y a\ b) = \text{get}_x a) \wedge (\forall a\ b. \text{get}_y (\text{put}_x a\ b) = \text{get}_y a)$   
**by** (*simp add: assms(3) lens-indep.lens-put-irr2 lens-indep-comm*)  
**then show**  $b_1 = b_2$   
**by** (*metis a assms(1) assms(2) pr-var-def var.rep-eq vwb-lens.source-determination vwb-lens-def*  
*wb-lens-def weak-lens.put-get*)  
**qed**

**lemma** *usubst-upd-var-id* [*usubst*]:  
 $\text{vwb-lens}\ x \Rightarrow [x \mapsto_s \text{var}\ x] = \text{id}$   
**apply** (*simp add: subst-upd-uvar-def*)  
**apply** (*transfer*)  
**apply** (*rule ext*)  
**apply** (*auto*)  
**done**

**lemma** *usubst-upd-pr-var-id* [*usubst*]:  
 $\text{vwb-lens}\ x \Rightarrow [x \mapsto_s \text{var}\ (\text{pr-var}\ x)] = \text{id}$   
**apply** (*simp add: subst-upd-uvar-def pr-var-def*)  
**apply** (*transfer*)  
**apply** (*rule ext*)  
**apply** (*auto*)  
**done**

**lemma** *usubst-upd-comm-dash* [*usubst*]:  
**fixes**  $x :: ('a \Rightarrow 'a)$   
**shows**  $\sigma(\$x' \mapsto_s v, \$x \mapsto_s u) = \sigma(\$x \mapsto_s u, \$x' \mapsto_s v)$   
**using** *out-in-indep usubst-upd-comm* **by** *blast*

**lemma** *subst-upd-lens-plus* [*usubst*]:  
 $\text{subst-upd}\ \sigma\ (x +_L y) \ll(u, v)\gg = \sigma(y \mapsto_s \ll v \gg, x \mapsto_s \ll u \gg)$   
**by** (*simp add: lens-defs uepr-defs subst-upd-uvar-def, transfer, auto*)

**lemma** *subst-upd-in-lens-plus* [*usubst*]:  
 $\text{subst-upd}\ \sigma\ (\text{ivar}\ (x +_L y)) \ll(u, v)\gg = \sigma(\$y \mapsto_s \ll v \gg, \$x \mapsto_s \ll u \gg)$

by (simp add: lens-defs uexpr-defs subst-upd-uvar-def, transfer, auto simp add: prod.case-eq-if)

**lemma** *subst-upd-out-lens-plus* [usubst]:

$\text{subst-upd } \sigma \text{ (ovar } (x +_L y)) \ll (u, v) \gg = \sigma(\$y' \mapsto_s \ll v \gg, \$x' \mapsto_s \ll u \gg)$

by (simp add: lens-defs uexpr-defs subst-upd-uvar-def, transfer, auto simp add: prod.case-eq-if)

**lemma** *usubst-lookup-upd-indep* [usubst]:

assumes  $\text{mwb-lens } x \bowtie y$

shows  $\langle \sigma(y \mapsto_s v) \rangle_s x = \langle \sigma \rangle_s x$

using *assms*

by (simp add: subst-upd-uvar-def, transfer, simp)

**lemma** *subst-upd-plus* [usubst]:

$x \bowtie y \implies \text{subst-upd } s \text{ (} x +_L y \text{)} e = s(x \mapsto_s \pi_1(e), y \mapsto_s \pi_2(e))$

by (simp add: subst-upd-uvar-def lens-defs, transfer, auto simp add: fun-eq-iff prod.case-eq-if lens-indep-comm)

If a variable is unrestricted in a substitution then it's application has no effect.

**lemma** *usubst-apply-unrest* [usubst]:

$\ll \text{vwb-lens } x; x \# \sigma \gg \implies \langle \sigma \rangle_s x = \text{var } x$

by (simp add: unrest-usubst-def, transfer, auto simp add: fun-eq-iff, metis vwb-lens-wb wb-lens.get-put wb-lens-weak weak-lens.put-get)

There follows various laws about deleting variables from a substitution.

**lemma** *subst-del-id* [usubst]:

$\text{vwb-lens } x \implies \text{id } -_s x = \text{id}$

by (simp add: subst-del-def subst-upd-uvar-def pr-var-def, transfer, auto)

**lemma** *subst-del-upd-same* [usubst]:

$\text{mwb-lens } x \implies \sigma(x \mapsto_s v) -_s x = \sigma -_s x$

by (simp add: subst-del-def subst-upd-uvar-def)

**lemma** *subst-del-upd-diff* [usubst]:

$x \bowtie y \implies \sigma(y \mapsto_s v) -_s x = (\sigma -_s x)(y \mapsto_s v)$

by (simp add: subst-del-def subst-upd-uvar-def lens-indep-comm)

If a variable is unrestricted in an expression, then any substitution of that variable has no effect on the expression .

**lemma** *subst-unrest* [usubst]:  $x \# P \implies \sigma(x \mapsto_s v) \dagger P = \sigma \dagger P$

by (simp add: subst-upd-uvar-def, transfer, auto)

**lemma** *subst-unrest-2* [usubst]:

fixes  $P :: ('a, 'α) \text{ uexpr}$

assumes  $x \# P \bowtie y$

shows  $\sigma(x \mapsto_s u, y \mapsto_s v) \dagger P = \sigma(y \mapsto_s v) \dagger P$

using *assms*

by (simp add: subst-upd-uvar-def, transfer, auto, metis lens-indep.lens-put-comm)

**lemma** *subst-unrest-3* [usubst]:

fixes  $P :: ('a, 'α) \text{ uexpr}$

assumes  $x \# P \bowtie y \bowtie z$

shows  $\sigma(x \mapsto_s u, y \mapsto_s v, z \mapsto_s w) \dagger P = \sigma(y \mapsto_s v, z \mapsto_s w) \dagger P$

using *assms*

by (simp add: subst-upd-uvar-def, transfer, auto, metis (no-types, hide-lams) lens-indep-comm)

**lemma** *subst-unrest-4* [usubst]:

**fixes**  $P :: ('a, 'α) \text{ uexpr}$   
**assumes**  $x \# P \ x \bowtie y \ x \bowtie z \ x \bowtie u$   
**shows**  $\sigma(x \mapsto_s e, y \mapsto_s f, z \mapsto_s g, u \mapsto_s h) \dagger P = \sigma(y \mapsto_s f, z \mapsto_s g, u \mapsto_s h) \dagger P$   
**using** *assms*  
**by** (*simp add: subst-upd-uvar-def, transfer, auto, metis (no-types, hide-lams) lens-indep-comm*)

**lemma** *subst-unrest-5* [*usubst*]:

**fixes**  $P :: ('a, 'α) \text{ uexpr}$   
**assumes**  $x \# P \ x \bowtie y \ x \bowtie z \ x \bowtie u \ x \bowtie v$   
**shows**  $\sigma(x \mapsto_s e, y \mapsto_s f, z \mapsto_s g, u \mapsto_s h, v \mapsto_s i) \dagger P = \sigma(y \mapsto_s f, z \mapsto_s g, u \mapsto_s h, v \mapsto_s i) \dagger P$   
**using** *assms*  
**by** (*simp add: subst-upd-uvar-def, transfer, auto, metis (no-types, hide-lams) lens-indep-comm*)

**lemma** *subst-compose-upd* [*usubst*]:  $x \# \sigma \implies \sigma \circ \varrho(x \mapsto_s v) = (\sigma \circ \varrho)(x \mapsto_s v)$

**by** (*simp add: subst-upd-uvar-def, transfer, auto simp add: unrest-usubst-def*)

Any substitution is a monotonic function.

**lemma** *subst-mono*: *mono* (*subst*  $\sigma$ )

**by** (*simp add: less-eq-uexpr.rep-eq mono-def subst.rep-eq*)

## 7.4 Substitution laws

We now prove the key laws that show how a substitution should be performed for every expression operator, including the core function operators, literals, variables, and the arithmetic operators. They are all added to the *usubst* theorem attribute so that we can apply them using the substitution tactic.

**lemma** *id-subst* [*usubst*]:  $id \dagger v = v$

**by** (*transfer, simp*)

**lemma** *subst-lit* [*usubst*]:  $\sigma \dagger \langle\!\langle v \rangle\!\rangle = \langle\!\langle v \rangle\!\rangle$

**by** (*transfer, simp*)

**lemma** *subst-var* [*usubst*]:  $\sigma \dagger \text{var } x = \langle\sigma\rangle_s x$

**by** (*transfer, simp*)

**lemma** *usubst-ulambda* [*usubst*]:  $\sigma \dagger (\lambda x \cdot P(x)) = (\lambda x \cdot \sigma \dagger P(x))$

**by** (*transfer, simp*)

**lemma** *unrest-usubst-del* [*unrest*]:  $\llbracket \text{vwb-lens } x; x \# (\langle\sigma\rangle_s x); x \# \sigma -_s x \rrbracket \implies x \# (\sigma \dagger P)$

**by** (*simp add: subst-del-def subst-upd-uvar-def unrest-uexpr-def unrest-usubst-def subst.rep-eq usubst-lookup.rep-eq (metis vwb-lens.put-eq)*)

We add the symmetric definition of input and output variables to substitution laws so that the variables are correctly normalised after substitution.

**lemma** *subst-uop* [*usubst*]:  $\sigma \dagger \text{uop } f \ v = \text{uop } f \ (\sigma \dagger v)$

**by** (*transfer, simp*)

**lemma** *subst-bop* [*usubst*]:  $\sigma \dagger \text{bop } f \ u \ v = \text{bop } f \ (\sigma \dagger u) \ (\sigma \dagger v)$

**by** (*transfer, simp*)

**lemma** *subst-trop* [*usubst*]:  $\sigma \dagger \text{trop } f \ u \ v \ w = \text{trop } f \ (\sigma \dagger u) \ (\sigma \dagger v) \ (\sigma \dagger w)$

**by** (*transfer, simp*)

**lemma** *subst-qtop* [*usubst*]:  $\sigma \dagger \text{qtop } f \ u \ v \ w \ x = \text{qtop } f \ (\sigma \dagger u) \ (\sigma \dagger v) \ (\sigma \dagger w) \ (\sigma \dagger x)$

by (transfer, simp)

**lemma** subst-case-prod [usubst]:  
 fixes  $P :: 'i \Rightarrow 'j \Rightarrow ('a, 'a) \text{ uexpr}$   
 shows  $\sigma \dagger \text{case-prod } (\lambda x y. P x y) v = \text{case-prod } (\lambda x y. \sigma \dagger P x y) v$   
 by (simp add: case-prod-beta')

**lemma** subst-plus [usubst]:  $\sigma \dagger (x + y) = \sigma \dagger x + \sigma \dagger y$   
 by (simp add: plus-uexpr-def subst-bop)

**lemma** subst-times [usubst]:  $\sigma \dagger (x * y) = \sigma \dagger x * \sigma \dagger y$   
 by (simp add: times-uexpr-def subst-bop)

**lemma** subst-mod [usubst]:  $\sigma \dagger (x \bmod y) = \sigma \dagger x \bmod \sigma \dagger y$   
 by (simp add: mod-uexpr-def usubst)

**lemma** subst-div [usubst]:  $\sigma \dagger (x \text{ div } y) = \sigma \dagger x \text{ div } \sigma \dagger y$   
 by (simp add: divide-uexpr-def usubst)

**lemma** subst-minus [usubst]:  $\sigma \dagger (x - y) = \sigma \dagger x - \sigma \dagger y$   
 by (simp add: minus-uexpr-def subst-bop)

**lemma** subst-uminus [usubst]:  $\sigma \dagger (- x) = - (\sigma \dagger x)$   
 by (simp add: uminus-uexpr-def subst-uop)

**lemma** usubst-sgn [usubst]:  $\sigma \dagger \text{sgn } x = \text{sgn } (\sigma \dagger x)$   
 by (simp add: sgn-uexpr-def subst-uop)

**lemma** usubst-abs [usubst]:  $\sigma \dagger \text{abs } x = \text{abs } (\sigma \dagger x)$   
 by (simp add: abs-uexpr-def subst-uop)

**lemma** subst-zero [usubst]:  $\sigma \dagger 0 = 0$   
 by (simp add: zero-uexpr-def subst-lit)

**lemma** subst-one [usubst]:  $\sigma \dagger 1 = 1$   
 by (simp add: one-uexpr-def subst-lit)

**lemma** subst-eq-upred [usubst]:  $\sigma \dagger (x =_u y) = (\sigma \dagger x =_u \sigma \dagger y)$   
 by (simp add: eq-upred-def usubst)

This laws shows the effect of applying one substitution after another – we simply use function composition to compose them.

**lemma** subst-subst [usubst]:  $\sigma \dagger \varrho \dagger e = (\varrho \circ \sigma) \dagger e$   
 by (transfer, simp)

The next law is similar, but shows how such a substitution is to be applied to every updated variable additionally.

**lemma** subst-upd-comp [usubst]:  
 fixes  $x :: ('a \Rightarrow 'a)$   
 shows  $\varrho(x \mapsto_s v) \circ \sigma = (\varrho \circ \sigma)(x \mapsto_s \sigma \dagger v)$   
 by (rule ext, simp add: uexpr-defs subst-upd-uvar-def, transfer, simp)

**lemma** subst-singleton:  
 fixes  $x :: ('a \Rightarrow 'a)$   
 assumes  $x \# \sigma$



shows  $\sigma(x \mapsto_s v) \dagger P = (\sigma \dagger P)[v/x]$   
 using *assms*  
 by (*simp add: usubst*)

lemmas *subst-to-singleton* = *subst-singleton id-subst*

## 7.5 Ordering substitutions

A simplification procedure to reorder substitutions maplets lexicographically by variable syntax

```
simproc-setup subst-order (subst-upd-uvar (subst-upd-uvar  $\sigma$   $x$   $u$ )  $y$   $v$ ) =
  <(fn - => fn ctxt => fn ct =>
    case (Thm.term-of ct) of
      Const (utp-subst.subst-upd-uvar, -) $ (Const (utp-subst.subst-upd-uvar, -) $  $s$  $  $x$  $  $u$ ) $  $y$  $  $v$ 
    => if (YXML.content-of (Syntax.string-of-term ctxt  $x$ ) > YXML.content-of (Syntax.string-of-term
ctxt  $y$ ))
      then SOME (mk-meta-eq @{thm usubst-upd-comm})
      else NONE |
    - => NONE)
  )
```

## 7.6 Unrestriction laws

These are the key unrestricted theorems for substitutions and expressions involving substitutions.

**lemma** *unrest-usubst-single* [*unrest*]:  
 $\llbracket \text{mwb-lens } x; x \# v \rrbracket \implies x \# P[v/x]$   
 by (*transfer, auto simp add: subst-upd-uvar-def unrest-uepr-def*)

**lemma** *unrest-usubst-id* [*unrest*]:  
 $\text{mwb-lens } x \implies x \# \text{id}$   
 by (*simp add: unrest-usubst-def*)

**lemma** *unrest-usubst-upd* [*unrest*]:  
 $\llbracket x \bowtie y; x \# \sigma; x \# v \rrbracket \implies x \# \sigma(y \mapsto_s v)$   
 by (*simp add: subst-upd-uvar-def unrest-usubst-def unrest-uepr.rep-eq lens-indep-comm*)

**lemma** *unrest-subst* [*unrest*]:  
 $\llbracket x \# P; x \# \sigma \rrbracket \implies x \# (\sigma \dagger P)$   
 by (*transfer, simp add: unrest-usubst-def*)

## 7.7 Conditional Substitution Laws

**lemma** *usubst-cond-upd-1* [*usubst*]:  
 $\sigma(x \mapsto_s u) \triangleleft b \triangleright_s \varrho(x \mapsto_s v) = (\sigma \triangleleft b \triangleright_s \varrho)(x \mapsto_s u \triangleleft b \triangleright v)$   
 by (*simp add: cond-subst-def subst-upd-uvar-def uepr-defs, transfer, auto*)

**lemma** *usubst-cond-upd-2* [*usubst*]:  
 $\llbracket \text{vwb-lens } x; x \# \varrho \rrbracket \implies \sigma(x \mapsto_s u) \triangleleft b \triangleright_s \varrho = (\sigma \triangleleft b \triangleright_s \varrho)(x \mapsto_s u \triangleleft b \triangleright \&x)$   
 by (*simp add: cond-subst-def subst-upd-uvar-def unrest-usubst-def uepr-defs, transfer*)  
 (*metis* (*full-types, hide-lams*) *id-apply pr-var-def subst-upd-uvar-def usubst-upd-pr-var-id var.rep-eq*)

**lemma** *usubst-cond-upd-3* [*usubst*]:  
 $\llbracket \text{vwb-lens } x; x \# \sigma \rrbracket \implies \sigma \triangleleft b \triangleright_s \varrho(x \mapsto_s v) = (\sigma \triangleleft b \triangleright_s \varrho)(x \mapsto_s \&x \triangleleft b \triangleright v)$   
 by (*simp add: cond-subst-def subst-upd-uvar-def unrest-usubst-def uepr-defs, transfer*)

(metis (full-types, hide-lams) id-apply pr-var-def subst-upd-uvar-def usubst-upd-pr-var-id var.rep-eq)

**lemma** *usubst-cond-id* [usubst]:

$\sigma \triangleleft b \triangleright_s \sigma = \sigma$

**by** (auto simp add: cond-subst-def)

## 7.8 Parallel Substitution Laws

**lemma** *par-subst-id* [usubst]:

$\llbracket \text{vwb-lens } A; \text{vwb-lens } B \rrbracket \implies \text{id } [A|B]_s \text{id} = \text{id}$

**by** (simp add: par-subst-def id-def)

**lemma** *par-subst-left-empty* [usubst]:

$\llbracket \text{vwb-lens } A \rrbracket \implies \sigma [\emptyset|A]_s \varrho = \text{id } [\emptyset|A]_s \varrho$

**by** (simp add: par-subst-def pr-var-def)

**lemma** *par-subst-right-empty* [usubst]:

$\llbracket \text{vwb-lens } A \rrbracket \implies \sigma [A|\emptyset]_s \varrho = \sigma [A|\emptyset]_s \text{id}$

**by** (simp add: par-subst-def pr-var-def)

**lemma** *par-subst-comm*:

$\llbracket A \bowtie B \rrbracket \implies \sigma [A|B]_s \varrho = \varrho [B|A]_s \sigma$

**by** (simp add: par-subst-def lens-override-def lens-indep-comm)

**lemma** *par-subst-upd-left-in* [usubst]:

$\llbracket \text{vwb-lens } A; A \bowtie B; x \subseteq_L A \rrbracket \implies \sigma(x \mapsto_s v) [A|B]_s \varrho = (\sigma [A|B]_s \varrho)(x \mapsto_s v)$

**by** (simp add: par-subst-def subst-upd-uvar-def lens-override-put-right-in)

(simp add: lens-indep-comm lens-override-def sublens-pres-indep)

**lemma** *par-subst-upd-left-out* [usubst]:

$\llbracket \text{vwb-lens } A; x \bowtie A \rrbracket \implies \sigma(x \mapsto_s v) [A|B]_s \varrho = (\sigma [A|B]_s \varrho)$

**by** (simp add: par-subst-def subst-upd-uvar-def lens-override-put-right-out)

**lemma** *par-subst-upd-right-in* [usubst]:

$\llbracket \text{vwb-lens } B; A \bowtie B; x \subseteq_L B \rrbracket \implies \sigma [A|B]_s \varrho(x \mapsto_s v) = (\sigma [A|B]_s \varrho)(x \mapsto_s v)$

**using** lens-indep-sym par-subst-comm par-subst-upd-left-in **by** fastforce

**lemma** *par-subst-upd-right-out* [usubst]:

$\llbracket \text{vwb-lens } B; A \bowtie B; x \bowtie B \rrbracket \implies \sigma [A|B]_s \varrho(x \mapsto_s v) = (\sigma [A|B]_s \varrho)$

**by** (simp add: par-subst-comm par-subst-upd-left-out)

**end**

## 8 UTP Tactics

```
theory utp-tactics
imports
  utp-expr utp-unrest utp-usedby
keywords update-ueexpr-rep-eq-thms :: thy-decl
begin

declare image-comp [simp]
```

In this theory, we define several automatic proof tactics that use transfer techniques to re-interpret proof goals about UTP predicates and relations in terms of pure HOL conjectures. The fundamental tactics to achieve this are *pred-simp* and *rel-simp*; a more detailed explanation of their behaviour is given below. The tactics can be given optional arguments to fine-tune their behaviour. By default, they use a weaker but faster form of transfer using rewriting; the option *robust*, however, forces them to use the slower but more powerful transfer of Isabelle’s lifting package. A second option *no-interp* suppresses the re-interpretation of state spaces in order to eradicate record for tuple types prior to automatic proof.

In addition to *pred-simp* and *rel-simp*, we also provide the tactics *pred-auto* and *rel-auto*, as well as *pred-blast* and *rel-blast*; they, in essence, sequence the simplification tactics with the methods *auto* and *blast*, respectively.

### 8.1 Theorem Attributes

The following named attributes have to be introduced already here since our tactics must be able to see them. Note that we do not want to import the theories *utp-pred* and *utp-rel* here, so that both can potentially already make use of the tactics we define in this theory.

```
named-theorems upred-defs upred definitional theorems
named-theorems urel-defs urel definitional theorems
```

### 8.2 Generic Methods

We set up several automatic tactics that recast theorems on UTP predicates into equivalent HOL predicates, eliminating artefacts of the mechanisation as much as this is possible. Our approach is first to unfold all relevant definition of the UTP predicate model, then perform a transfer, and finally simplify by using lens and variable definitions, the split laws of alphabet records, and interpretation laws to convert record-based state spaces into products. The definition of the respective methods is facilitated by the Eisbach tool: we define generic methods that are parametrised by the tactics used for transfer, interpretation and subsequent automatic proof. Note that the tactics only apply to the head goal.

#### Generic Predicate Tactics

```
method gen-pred-tac methods transfer-tac interp-tac prove-tac = (
  ((unfold upred-defs) [1])?;
  (transfer-tac),
  (simp add: fun-eq-iff
    lens-defs upred-defs alpha-splits Product-Type.split-beta)?,
  (interp-tac)?);
(prove-tac)
```

#### Generic Relational Tactics

```

method gen-rel-tac methods transfer-tac interp-tac prove-tac = (
  ((unfold upred-defs urel-defs) [1])?;
  (transfer-tac),
  (simp add: fun-eq-iff relcomp-unfold OO-def
    lens-defs upred-defs alpha-splits Product-Type.split-beta)?,
  (interp-tac)?);
(prove-tac)

```

### 8.3 Transfer Tactics

Next, we define the component tactics used for transfer.

#### 8.3.1 Robust Transfer

Robust transfer uses the transfer method of the lifting package.

```

method slow-uexpr-transfer = (transfer)

```

#### 8.3.2 Faster Transfer

Fast transfer side-steps the use of the (*transfer*) method in favour of plain rewriting with the underlying *rep-eq-...* laws of lifted definitions. For moderately complex terms, surprisingly, the transfer step turned out to be a bottle-neck in some proofs; we observed that faster transfer resulted in a speed-up of approximately 30% when building the UTP theory heaps. On the downside, tactics using faster transfer do not always work but merely in about 95% of the cases. The approach typically works well when proving predicate equalities and refinements conjectures.

A known limitation is that the faster tactic, unlike lifting transfer, does not turn free variables into meta-quantified ones. This can, in some cases, interfere with the interpretation step and cause subsequent application of automatic proof tactics to fail. A fix is in progress [TODO].

**Attribute Setup** We first configure a dynamic attribute *uexpr-rep-eq-thms* to automatically collect all *rep-eq-* laws of lifted definitions on the *uexpr* type.

**ML-file** *uexpr-rep-eq.ML*

```

setup (
  Global-Theory.add-thms-dynamic (@{binding uexpr-rep-eq-thms},
    uexpr-rep-eq.get-uexpr-rep-eq-thms o Context.theory-of)
)

```

We next configure a command **update-uexpr-rep-eq-thms** in order to update the content of the *uexpr-rep-eq-thms* attribute. Although the relevant theorems are collected automatically, for efficiency reasons, the user has to manually trigger the update process. The command must hence be executed whenever new lifted definitions for type *uexpr* are created. The updating mechanism uses **find-theorems** under the hood.

```

ML (
  Outer-Syntax.command @{command-keyword update-uexpr-rep-eq-thms}
    reread and update content of the uexpr-rep-eq-thms attribute
    (Scan.succeed (Toplevel.theory uexpr-rep-eq.read-uexpr-rep-eq-thms));
)

```

**update-uexpr-rep-eq-thms** — Read *uexpr-rep-eq-thms* here.

Lastly, we require several named-theorem attributes to record the manual transfer laws and extra simplifications, so that the user can dynamically extend them in child theories.

**named-theorems** *uexpr-transfer-laws* *uexpr transfer laws*

**declare** *uexpr-eq-iff* [*uexpr-transfer-laws*]

**named-theorems** *uexpr-transfer-extra* *extra simplifications for uexpr transfer*

**declare** *unrest-uexpr.rep-eq* [*uexpr-transfer-extra*]

*usedBy-uexpr.rep-eq* [*uexpr-transfer-extra*]

*utp-expr.numeral-uexpr-rep-eq* [*uexpr-transfer-extra*]

*utp-expr.less-eq-uexpr.rep-eq* [*uexpr-transfer-extra*]

*Abs-uexpr-inverse* [*simplified, uexpr-transfer-extra*]

*Rep-uexpr-inverse* [*uexpr-transfer-extra*]

**Tactic Definition** We have all ingredients now to define the fast transfer tactic as a single simplification step.

**method** *fast-uexpr-transfer* =

(*simp add: uexpr-transfer-laws uexpr-rep-eq-thms uexpr-transfer-extra*)

## 8.4 Interpretation

The interpretation of record state spaces as products is done using the laws provided by the utility theory *Interp*. Note that this step can be suppressed by using the *no-interp* option.

**method** *uexpr-interp-tac* = (*simp add: lens-interp-laws*)?

## 8.5 User Tactics

In this section, we finally set-up the six user tactics: *pred-simp*, *rel-simp*, *pred-auto*, *rel-auto*, *pred-blast* and *rel-blast*. For this, we first define the proof strategies that are to be applied *after* the transfer steps.

**method** *utp-simp-tac* = (*clarsimp*)?

**method** *utp-auto-tac* = ((*clarsimp*)?; *auto*)

**method** *utp-blast-tac* = ((*clarsimp*)?; *blast*)

The ML file below provides ML constructor functions for tactics that process arguments suitable and invoke the generic methods *gen-pred-tac* and *gen-rel-tac* with suitable arguments.

**ML-file** *utp-tactics.ML*

Finally, we execute the relevant outer commands for method setup. Sadly, this cannot be done at the level of Eisbach since the latter does not provide a convenient mechanism to process symbolic flags as arguments. It may be worth to put in a feature request with the developers of the Eisbach tool.

**method-setup** *pred-simp* = (

(*Scan.lift UTP-Tactics.scan-args*) >>

(*fn args => fn ctxt =>*

*let val prove-tac = Basic-Tactics.utp-simp-tac in*

*(UTP-Tactics.inst-gen-pred-tac args prove-tac ctxt)*

*end*)

)

```

method-setup rel-simp = ⟨
  (Scan.lift UTP-Tactics.scan-args) >>
  (fn args => fn ctxt =>
    let val prove-tac = Basic-Tactics.utp-simp-tac in
    (UTP-Tactics.inst-gen-rel-tac args prove-tac ctxt)
  end)
⟩

method-setup pred-auto = ⟨
  (Scan.lift UTP-Tactics.scan-args) >>
  (fn args => fn ctxt =>
    let val prove-tac = Basic-Tactics.utp-auto-tac in
    (UTP-Tactics.inst-gen-pred-tac args prove-tac ctxt)
  end)
⟩

method-setup rel-auto = ⟨
  (Scan.lift UTP-Tactics.scan-args) >>
  (fn args => fn ctxt =>
    let val prove-tac = Basic-Tactics.utp-auto-tac in
    (UTP-Tactics.inst-gen-rel-tac args prove-tac ctxt)
  end)
⟩

method-setup pred-blast = ⟨
  (Scan.lift UTP-Tactics.scan-args) >>
  (fn args => fn ctxt =>
    let val prove-tac = Basic-Tactics.utp-blast-tac in
    (UTP-Tactics.inst-gen-pred-tac args prove-tac ctxt)
  end)
⟩

method-setup rel-blast = ⟨
  (Scan.lift UTP-Tactics.scan-args) >>
  (fn args => fn ctxt =>
    let val prove-tac = Basic-Tactics.utp-blast-tac in
    (UTP-Tactics.inst-gen-rel-tac args prove-tac ctxt)
  end)
⟩

```

Simpler, one-shot versions of the above tactics, but without the possibility of dynamic arguments.

```

method rel-simp'
  uses simp
  = (simp add: upred-defs urel-defs lens-defs prod.case-eq-if relcomp-unfold uexpr-transfer-laws uexpr-transfer-extra
    uexpr-rep-eq-thms simp)

method rel-auto'
  uses simp intro elim dest
  = (auto intro: intro elim: elim dest: dest simp add: upred-defs urel-defs lens-defs relcomp-unfold
    uexpr-transfer-laws uexpr-transfer-extra uexpr-rep-eq-thms simp)

method rel-blast'
  uses simp intro elim dest

```

$$= (rel-simp' \ simp: \ simp, \ blast \ intro: \ intro \ elim: \ elim \ dest: \ dest)$$

end

## 9 Meta-level Substitution

```
theory utp-meta-subst
imports utp-subst utp-tactics
begin
```

Meta substitution substitutes a HOL variable in a UTP expression for another UTP expression. It is analogous to UTP substitution, but acts on functions.

**lift-definition**  $msubst :: ('b \Rightarrow ('a, 'a) uexpr) \Rightarrow ('b, 'a) uexpr \Rightarrow ('a, 'a) uexpr$   
**is**  $\lambda F v b. F (v b) b$ .

**update-uexpr-rep-eq-thms** — Reread *rep-eq* theorems.

**syntax**

$-msubst \quad :: \text{logic} \Rightarrow \text{pttrn} \Rightarrow \text{logic} \Rightarrow \text{logic} \ ((-\!\!\rightarrow\!-) \ [990,0,0] \ 991)$

**translations**

$-msubst \ P \ x \ v == \text{CONST } msubst \ (\lambda x. P) \ v$

**lemma** *msubst-lit* [usubst]:  $\ll x \gg \ll x \rightarrow v \gg = v$   
**by** (*pred-auto*)

**lemma** *msubst-const* [usubst]:  $P \ll x \rightarrow v \gg = P$   
**by** (*pred-auto*)

**lemma** *msubst-pair* [usubst]:  $(P \ x \ y) \ll (x, y) \rightarrow (e, f) \gg_u = (P \ x \ y) \ll x \rightarrow e \gg \ll y \rightarrow f \gg$   
**by** (*rel-auto*)

**lemma** *msubst-lit-2-1* [usubst]:  $\ll x \gg \ll (x, y) \rightarrow (u, v) \gg_u = u$   
**by** (*pred-auto*)

**lemma** *msubst-lit-2-2* [usubst]:  $\ll y \gg \ll (x, y) \rightarrow (u, v) \gg_u = v$   
**by** (*pred-auto*)

**lemma** *msubst-lit'* [usubst]:  $\ll y \gg \ll x \rightarrow v \gg = \ll y \gg$   
**by** (*pred-auto*)

**lemma** *msubst-lit'-2* [usubst]:  $\ll z \gg \ll (x, y) \rightarrow v \gg = \ll z \gg$   
**by** (*pred-auto*)

**lemma** *msubst-uop* [usubst]:  $(uop \ f \ (v \ x)) \ll x \rightarrow u \gg = uop \ f \ ((v \ x) \ll x \rightarrow u \gg)$   
**by** (*rel-auto*)

**lemma** *msubst-uop-2* [usubst]:  $(uop \ f \ (v \ x \ y)) \ll (x, y) \rightarrow u \gg = uop \ f \ ((v \ x \ y) \ll (x, y) \rightarrow u \gg)$   
**by** (*pred-simp, pred-simp*)

**lemma** *msubst-bop* [usubst]:  $(bop \ f \ (v \ x) \ (w \ x)) \ll x \rightarrow u \gg = bop \ f \ ((v \ x) \ll x \rightarrow u \gg) \ ((w \ x) \ll x \rightarrow u \gg)$   
**by** (*rel-auto*)

**lemma** *msubst-bop-2* [usubst]:  $(bop \ f \ (v \ x \ y) \ (w \ x \ y)) \ll (x, y) \rightarrow u \gg = bop \ f \ ((v \ x \ y) \ll (x, y) \rightarrow u \gg) \ ((w \ x \ y) \ll (x, y) \rightarrow u \gg)$   
**by** (*pred-simp, pred-simp*)



```

lemma msubst-var [usubst]:
  (utp-expr.var x) $\llbracket y \rightarrow u \rrbracket$  = utp-expr.var x
  by (pred-simp)

lemma msubst-var-2 [usubst]:
  (utp-expr.var x) $\llbracket (y, z) \rightarrow u \rrbracket$  = utp-expr.var x
  by (pred-simp)+

lemma msubst-unrest [unrest]:  $\llbracket \bigwedge v. x \# P(v); x \# k \rrbracket \Longrightarrow x \# P(v) \llbracket v \rightarrow k \rrbracket$ 
  by (pred-auto)

end

```

## 10 Alphabetised Predicates

```

theory utp-pred
imports
  utp-expr-funcs
  utp-subst
  utp-meta-subst
  utp-tactics
begin

```

In this theory we begin to create an Isabelle version of the alphabetised predicate calculus that is described in Chapter 1 of the UTP book [22].

### 10.1 Predicate type and syntax

An alphabetised predicate is simply a boolean valued expression.

```

type-synonym 'α upred = (bool, 'α) uexpr

```

```

translations
  (type) 'α upred <= (type) (bool, 'α) uexpr

```

We want to remain as close as possible to the mathematical UTP syntax, but also want to be conservative with HOL. For this reason we chose not to steal syntax from HOL, but where possible use polymorphism to allow selection of the appropriate operator (UTP vs. HOL). Thus we will first remove the standard syntax for conjunction, disjunction, and negation, and replace these with adhoc overloaded definitions. We similarly use polymorphic constants for the other predicate calculus operators.

```

purge-notation
  conj (infixr  $\wedge$  35) and
  disj (infixr  $\vee$  30) and
  Not ( $\neg$  - [40] 40)

```

```

consts
  uttrue :: 'a (true)
  ufalse :: 'a (false)
  uconj :: 'a  $\Rightarrow$  'a  $\Rightarrow$  'a (infixr  $\wedge$  35)
  udisj :: 'a  $\Rightarrow$  'a  $\Rightarrow$  'a (infixr  $\vee$  30)
  uimpl :: 'a  $\Rightarrow$  'a  $\Rightarrow$  'a (infixr  $\Rightarrow$  25)
  uiff :: 'a  $\Rightarrow$  'a  $\Rightarrow$  'a (infixr  $\Leftrightarrow$  25)
  unot :: 'a  $\Rightarrow$  'a ( $\neg$  - [40] 40)

```

$uex :: ('a \Rightarrow 'α) \Rightarrow 'p \Rightarrow 'p$   
 $uall :: ('a \Rightarrow 'α) \Rightarrow 'p \Rightarrow 'p$   
 $ushEx :: ['a \Rightarrow 'p] \Rightarrow 'p$   
 $ushAll :: ['a \Rightarrow 'p] \Rightarrow 'p$

#### ad hoc-overloading

$uconj$  *conj* **and**  
 $udisj$  *disj* **and**  
 $unot$  *Not*

We set up two versions of each of the quantifiers:  $uex$  /  $uall$  and  $ushEx$  /  $ushAll$ . The former pair allows quantification of UTP variables, whilst the latter allows quantification of HOL variables in concert with the literal expression constructor  $\ll x \gg$ . Both varieties will be needed at various points. Syntactically they are distinguished by a boldface quantifier for the HOL versions (achieved by the "bold" escape in Isabelle).

#### nonterminal *idt-list*

#### syntax

$-idt-el :: idt \Rightarrow idt-list \ (-)$   
 $-idt-list :: idt \Rightarrow idt-list \Rightarrow idt-list \ ((-, / -) [0, 1])$   
 $-uex :: salpha \Rightarrow logic \Rightarrow logic \ (\exists \ - \ - \ [0, 10] \ 10)$   
 $-uall :: salpha \Rightarrow logic \Rightarrow logic \ (\forall \ - \ - \ [0, 10] \ 10)$   
 $-ushEx :: pttrn \Rightarrow logic \Rightarrow logic \ (\exists \ - \ - \ [0, 10] \ 10)$   
 $-ushAll :: pttrn \Rightarrow logic \Rightarrow logic \ (\forall \ - \ - \ [0, 10] \ 10)$   
 $-ushBEx :: pttrn \Rightarrow logic \Rightarrow logic \Rightarrow logic \ (\exists \ - \in \ - \ - \ [0, 0, 10] \ 10)$   
 $-ushBAll :: pttrn \Rightarrow logic \Rightarrow logic \Rightarrow logic \ (\forall \ - \in \ - \ - \ [0, 0, 10] \ 10)$   
 $-ushGAll :: pttrn \Rightarrow logic \Rightarrow logic \Rightarrow logic \ (\forall \ - \mid \ - \ - \ [0, 0, 10] \ 10)$   
 $-ushGtAll :: idt \Rightarrow logic \Rightarrow logic \Rightarrow logic \ (\forall \ - \> \ - \ - \ [0, 0, 10] \ 10)$   
 $-ushLtAll :: idt \Rightarrow logic \Rightarrow logic \Rightarrow logic \ (\forall \ - \< \ - \ - \ [0, 0, 10] \ 10)$   
 $-uvar-res :: logic \Rightarrow salpha \Rightarrow logic \ (\mathbf{infixl} \ \vdash_v \ 90)$

#### translations

$-uex \ x \ P \quad \quad \quad == \text{CONST } uex \ x \ P$   
 $-uex \ (-salphaset \ (-salphamk \ (x +_L y))) \ P \leq -uex \ (x +_L y) \ P$   
 $-uall \ x \ P \quad \quad \quad == \text{CONST } uall \ x \ P$   
 $-uall \ (-salphaset \ (-salphamk \ (x +_L y))) \ P \leq -uall \ (x +_L y) \ P$   
 $-ushEx \ x \ P \quad \quad \quad == \text{CONST } ushEx \ (\lambda x. P)$   
 $\exists \ x \in A \cdot P \quad \quad \quad \Rightarrow \exists \ x \cdot \ll x \gg \in_u A \wedge P$   
 $-ushAll \ x \ P \quad \quad \quad == \text{CONST } ushAll \ (\lambda x. P)$   
 $\forall \ x \in A \cdot P \quad \quad \quad \Rightarrow \forall \ x \cdot \ll x \gg \in_u A \Rightarrow P$   
 $\forall \ x \mid P \cdot Q \quad \quad \quad \Rightarrow \forall \ x \cdot P \Rightarrow Q$   
 $\forall \ x > y \cdot P \quad \quad \quad \Rightarrow \forall \ x \cdot \ll x \gg >_u y \Rightarrow P$   
 $\forall \ x < y \cdot P \quad \quad \quad \Rightarrow \forall \ x \cdot \ll x \gg <_u y \Rightarrow P$

## 10.2 Predicate operators

We chose to maximally reuse definitions and laws built into HOL. For this reason, when introducing the core operators we proceed by lifting operators from the polymorphic algebraic hierarchy of HOL. Thus the initial definitions take place in the context of type class instantiations. We first introduce our own class called *refine* that will add the refinement operator syntax to the HOL partial order class.

**class** *refine* = *order*

**abbreviation** *refineBy* ::  $'a :: refine \Rightarrow 'a \Rightarrow bool$  (**infix**  $\sqsubseteq$  50) **where**

$P \sqsubseteq Q \equiv \text{less-eq } Q \ P$

Since, on the whole, lattices in UTP are the opposite way up to the standard definitions in HOL, we syntactically invert the lattice operators. This is the one exception where we do steal HOL syntax, but I think it makes sense for UTP. Indeed we make this inversion for all of the lattice operators.

**purge-notation** *Lattices.inf* (**infixl**  $\sqcap$  70)  
**notation** *Lattices.inf* (**infixl**  $\sqcup$  70)  
**purge-notation** *Lattices.sup* (**infixl**  $\sqcup$  65)  
**notation** *Lattices.sup* (**infixl**  $\sqcap$  65)

**purge-notation** *Inf* ( $\sqcap$  - [900] 900)  
**notation** *Inf* ( $\sqcup$  - [900] 900)  
**purge-notation** *Sup* ( $\sqcup$  - [900] 900)  
**notation** *Sup* ( $\sqcap$  - [900] 900)

**purge-notation** *Orderings.bot* ( $\perp$ )  
**notation** *Orderings.bot* ( $\top$ )  
**purge-notation** *Orderings.top* ( $\top$ )  
**notation** *Orderings.top* ( $\perp$ )

**purge-syntax**

-INF1 :: *pttrns*  $\Rightarrow$  'b  $\Rightarrow$  'b (( $\exists \sqcap$  -./ -) [0, 10] 10)  
-INF :: *pttrn*  $\Rightarrow$  'a set  $\Rightarrow$  'b  $\Rightarrow$  'b (( $\exists \sqcap$  - $\in$  -./ -) [0, 0, 10] 10)  
-SUP1 :: *pttrns*  $\Rightarrow$  'b  $\Rightarrow$  'b (( $\exists \sqcup$  -./ -) [0, 10] 10)  
-SUP :: *pttrn*  $\Rightarrow$  'a set  $\Rightarrow$  'b  $\Rightarrow$  'b (( $\exists \sqcup$  - $\in$  -./ -) [0, 0, 10] 10)

**syntax**

-INF1 :: *pttrns*  $\Rightarrow$  'b  $\Rightarrow$  'b (( $\exists \sqcup$  -./ -) [0, 10] 10)  
-INF :: *pttrn*  $\Rightarrow$  'a set  $\Rightarrow$  'b  $\Rightarrow$  'b (( $\exists \sqcup$  - $\in$  -./ -) [0, 0, 10] 10)  
-SUP1 :: *pttrns*  $\Rightarrow$  'b  $\Rightarrow$  'b (( $\exists \sqcap$  -./ -) [0, 10] 10)  
-SUP :: *pttrn*  $\Rightarrow$  'a set  $\Rightarrow$  'b  $\Rightarrow$  'b (( $\exists \sqcap$  - $\in$  -./ -) [0, 0, 10] 10)

We trivially instantiate our refinement class

**instance** *uexpr* :: (*order*, *type*) *refine* ..

— Configure transfer law for refinement for the fast relational tactics.

**theorem** *upred-ref-iff* [*uexpr-transfer-laws*]:

$(P \sqsubseteq Q) = (\forall b. \llbracket Q \rrbracket_e b \longrightarrow \llbracket P \rrbracket_e b)$

**apply** (*transfer*)

**apply** (*clarsimp*)

**done**

Next we introduce the lattice operators, which is again done by lifting.

**instantiation** *uexpr* :: (*lattice*, *type*) *lattice*

**begin**

**lift-definition** *sup-uexpr* :: ('a, 'b) *uexpr*  $\Rightarrow$  ('a, 'b) *uexpr*  $\Rightarrow$  ('a, 'b) *uexpr*

**is**  $\lambda P \ Q \ A. \text{Lattices.sup } (P \ A) \ (Q \ A)$  .

**lift-definition** *inf-uexpr* :: ('a, 'b) *uexpr*  $\Rightarrow$  ('a, 'b) *uexpr*  $\Rightarrow$  ('a, 'b) *uexpr*

**is**  $\lambda P \ Q \ A. \text{Lattices.inf } (P \ A) \ (Q \ A)$  .

**instance**

**by** (*intro-classes*) (*transfer*, *auto*)+

**end**

```

instantiation uexpr :: (bounded-lattice, type) bounded-lattice
begin
  lift-definition bot-uexpr :: ('a, 'b) uexpr is  $\lambda A. \text{Orderings.bot}$  .
  lift-definition top-uexpr :: ('a, 'b) uexpr is  $\lambda A. \text{Orderings.top}$  .
instance
  by (intro-classes) (transfer, auto)+
end

```

```

lemma top-uexpr-rep-eq [simp]:
   $\llbracket \text{Orderings.bot} \rrbracket_e b = \text{False}$ 
  by (transfer, auto)

```

```

lemma bot-uexpr-rep-eq [simp]:
   $\llbracket \text{Orderings.top} \rrbracket_e b = \text{True}$ 
  by (transfer, auto)

```

```

instance uexpr :: (distrib-lattice, type) distrib-lattice
  by (intro-classes) (transfer, rule ext, auto simp add: sup-inf-distrib1)

```

Finally we show that predicates form a Boolean algebra (under the lattice operators), a complete lattice, a completely distribute lattice, and a complete boolean algebra. This equip us with a very complete theory for basic logical propositions.

```

instance uexpr :: (boolean-algebra, type) boolean-algebra
  apply (intro-classes, unfold uexpr-defs; transfer, rule ext)
  apply (simp-all add: sup-inf-distrib1 diff-eq)
  done

```

```

instantiation uexpr :: (complete-lattice, type) complete-lattice
begin
  lift-definition Inf-uexpr :: ('a, 'b) uexpr set  $\Rightarrow$  ('a, 'b) uexpr
  is  $\lambda PS A. \text{INF } P:PS. P(A)$  .
  lift-definition Sup-uexpr :: ('a, 'b) uexpr set  $\Rightarrow$  ('a, 'b) uexpr
  is  $\lambda PS A. \text{SUP } P:PS. P(A)$  .
instance
  by (intro-classes)
  (transfer, auto intro: INF-lower SUP-upper simp add: INF-greatest SUP-least)+
end

```

```

instance uexpr :: (complete-distrib-lattice, type) complete-distrib-lattice
  by (intro-classes; transfer; auto simp add: INF-SUP-set)

```

```

instance uexpr :: (complete-boolean-algebra, type) complete-boolean-algebra ..

```

From the complete lattice, we can also define and give syntax for the fixed-point operators. Like the lattice operators, these are reversed in UTP.

```

syntax
  -mu :: pttrn  $\Rightarrow$  logic  $\Rightarrow$  logic ( $\mu$  - - [0, 10] 10)
  -nu :: pttrn  $\Rightarrow$  logic  $\Rightarrow$  logic ( $\nu$  - - [0, 10] 10)

```

```

notation gfp ( $\mu$ )
notation lfp ( $\nu$ )

```

```

translations
   $\nu X \cdot P == \text{CONST lfp } (\lambda X. P)$ 
   $\mu X \cdot P == \text{CONST gfp } (\lambda X. P)$ 

```

With the lattice operators defined, we can proceed to give definitions for the standard predicate operators in terms of them.

**definition**  $true-upred = (Orderings.top :: 'α upred)$

**definition**  $false-upred = (Orderings.bot :: 'α upred)$

**definition**  $conj-upred = (Lattices.inf :: 'α upred ⇒ 'α upred ⇒ 'α upred)$

**definition**  $disj-upred = (Lattices.sup :: 'α upred ⇒ 'α upred ⇒ 'α upred)$

**definition**  $not-upred = (uminus :: 'α upred ⇒ 'α upred)$

**definition**  $diff-upred = (minus :: 'α upred ⇒ 'α upred ⇒ 'α upred)$

**abbreviation**  $Conj-upred :: 'α upred set ⇒ 'α upred (\bigwedge - [900] 900)$  **where**  
 $\bigwedge A \equiv \bigcap A$

**abbreviation**  $Disj-upred :: 'α upred set ⇒ 'α upred (\bigvee - [900] 900)$  **where**  
 $\bigvee A \equiv \bigcup A$

**notation**

$conj-upred$  (**infixr**  $\wedge_p$  35) **and**

$disj-upred$  (**infixr**  $\vee_p$  30)

Perhaps slightly confusingly, the UTP infimum is the HOL supremum and vice-versa. This is because, again, in UTP the lattice is inverted due to the definition of refinement and a desire to have miracle at the top, and abort at the bottom.

**lift-definition**  $UINF :: ('a ⇒ 'α upred) ⇒ ('a ⇒ ('b::complete-lattice, 'α) uexpr) ⇒ ('b, 'α) uexpr$   
**is**  $\lambda P F b. Sup \{ \llbracket F x \rrbracket_e b \mid x. \llbracket P x \rrbracket_e b \} .$

**lift-definition**  $USUP :: ('a ⇒ 'α upred) ⇒ ('a ⇒ ('b::complete-lattice, 'α) uexpr) ⇒ ('b, 'α) uexpr$   
**is**  $\lambda P F b. Inf \{ \llbracket F x \rrbracket_e b \mid x. \llbracket P x \rrbracket_e b \} .$

**syntax**

$-USup \quad :: pttrn \Rightarrow logic \Rightarrow logic \quad (\bigwedge - \cdot - [0, 10] 10)$   
 $-USup \quad :: pttrn \Rightarrow logic \Rightarrow logic \quad (\bigcap - \cdot - [0, 10] 10)$   
 $-USup-mem :: pttrn \Rightarrow logic \Rightarrow logic \Rightarrow logic \quad (\bigwedge - \in \cdot - [0, 10] 10)$   
 $-USup-mem :: pttrn \Rightarrow logic \Rightarrow logic \Rightarrow logic \quad (\bigcap - \in \cdot - [0, 10] 10)$   
 $-USUP \quad :: pttrn \Rightarrow logic \Rightarrow logic \Rightarrow logic \quad (\bigwedge - | \cdot - [0, 0, 10] 10)$   
 $-USUP \quad :: pttrn \Rightarrow logic \Rightarrow logic \Rightarrow logic \quad (\bigcap - | \cdot - [0, 0, 10] 10)$   
 $-UInf \quad :: pttrn \Rightarrow logic \Rightarrow logic \quad (\bigvee - \cdot - [0, 10] 10)$   
 $-UInf \quad :: pttrn \Rightarrow logic \Rightarrow logic \quad (\bigcup - \cdot - [0, 10] 10)$   
 $-UInf-mem :: pttrn \Rightarrow logic \Rightarrow logic \Rightarrow logic \quad (\bigvee - \in \cdot - [0, 10] 10)$   
 $-UInf-mem :: pttrn \Rightarrow logic \Rightarrow logic \Rightarrow logic \quad (\bigcup - \in \cdot - [0, 10] 10)$   
 $-UINF \quad :: pttrn \Rightarrow logic \Rightarrow logic \Rightarrow logic \quad (\bigvee - | \cdot - [0, 10] 10)$   
 $-UINF \quad :: pttrn \Rightarrow logic \Rightarrow logic \Rightarrow logic \quad (\bigcup - | \cdot - [0, 10] 10)$

**translations**

$\bigcap x \mid P \cdot F \Rightarrow CONST UINF (\lambda x. P) (\lambda x. F)$   
 $\bigcap x \cdot F == \bigcap x \mid true \cdot F$   
 $\bigcap x \cdot F == \bigcap x \mid true \cdot F$   
 $\bigcap x \in A \cdot F \Rightarrow \bigcap x \mid \ll x \gg \in_u \ll A \gg \cdot F$   
 $\bigcap x \in A \cdot F \Leftarrow \bigcap x \mid \ll y \gg \in_u \ll A \gg \cdot F$   
 $\bigcap x \mid P \cdot F \Leftarrow CONST UINF (\lambda y. P) (\lambda x. F)$   
 $\bigcap x \mid P \cdot F(x) \Leftarrow CONST UINF (\lambda x. P) F$   
 $\bigcap x \mid P \cdot F \Rightarrow CONST USUP (\lambda x. P) (\lambda x. F)$   
 $\bigcap x \cdot F == \bigcap x \mid true \cdot F$   
 $\bigcap x \in A \cdot F \Rightarrow \bigcap x \mid \ll x \gg \in_u \ll A \gg \cdot F$   
 $\bigcap x \in A \cdot F \Leftarrow \bigcap x \mid \ll y \gg \in_u \ll A \gg \cdot F$   
 $\bigcap x \mid P \cdot F \Leftarrow CONST USUP (\lambda y. P) (\lambda x. F)$

$\sqcup \mid x \mid P \cdot F(x) \leq \text{CONST USUP } (\lambda x. P) F$

We also define the other predicate operators

**lift-definition** *impl* ::  $'\alpha \text{ upred} \Rightarrow '\alpha \text{ upred} \Rightarrow '\alpha \text{ upred}$  **is**  
 $\lambda P Q A. P A \longrightarrow Q A$  .

**lift-definition** *iff-upred* ::  $'\alpha \text{ upred} \Rightarrow '\alpha \text{ upred} \Rightarrow '\alpha \text{ upred}$  **is**  
 $\lambda P Q A. P A \longleftrightarrow Q A$  .

**lift-definition** *ex* ::  $('a \Longrightarrow '\alpha) \Rightarrow '\alpha \text{ upred} \Rightarrow '\alpha \text{ upred}$  **is**  
 $\lambda x P b. (\exists v. P(\text{put}_x b v))$  .

**lift-definition** *shEx* ::  $['\beta \Rightarrow '\alpha \text{ upred}] \Rightarrow '\alpha \text{ upred}$  **is**  
 $\lambda P A. \exists x. (P x) A$  .

**lift-definition** *all* ::  $('a \Longrightarrow '\alpha) \Rightarrow '\alpha \text{ upred} \Rightarrow '\alpha \text{ upred}$  **is**  
 $\lambda x P b. (\forall v. P(\text{put}_x b v))$  .

**lift-definition** *shAll* ::  $['\beta \Rightarrow '\alpha \text{ upred}] \Rightarrow '\alpha \text{ upred}$  **is**  
 $\lambda P A. \forall x. (P x) A$  .

We define the following operator which is dual of existential quantification. It hides the valuation of variables other than  $x$  through existential quantification.

**lift-definition** *var-res* ::  $'\alpha \text{ upred} \Rightarrow ('a \Longrightarrow '\alpha) \Rightarrow '\alpha \text{ upred}$  **is**  
 $\lambda P x b. \exists b'. P (b' \oplus_L b \text{ on } x)$  .

#### translations

*-uvar-res*  $P a \Rightarrow \text{CONST var-res } P a$

We have to add a  $u$  subscript to the closure operator as I don't want to override the syntax for HOL lists (we'll be using them later).

**lift-definition** *closure* ::  $'\alpha \text{ upred} \Rightarrow '\alpha \text{ upred } ([\cdot]_u)$  **is**  
 $\lambda P A. \forall A'. P A'$  .

**lift-definition** *taut* ::  $'\alpha \text{ upred} \Rightarrow \text{bool } ('-')$   
**is**  $\lambda P. \forall A. P A$  .

Configuration for UTP tactics

**update-uexpr-rep-eq-thms** — Reread *rep-eq* theorems.

**declare** *utp-pred.taut.rep-eq* [*upred-defs*]

#### ad hoc overloading

*uttrue* *true-upred* **and**  
*ufalse* *false-upred* **and**  
*unot* *not-upred* **and**  
*uconj* *conj-upred* **and**  
*udisj* *disj-upred* **and**  
*uimpl* *impl* **and**  
*uiff* *iff-upred* **and**  
*uex* *ex* **and**  
*uall* *all* **and**  
*ushEx* *shEx* **and**  
*ushAll* *shAll*

## syntax

-uneq ::  $logic \Rightarrow logic \Rightarrow logic$  (**infixl**  $\neq_u$  50)  
-unmem ::  $('a, 'α) uexpr \Rightarrow ('a\ set, 'α) uexpr \Rightarrow (bool, 'α) uexpr$  (**infix**  $\notin_u$  50)

## translations

$x \neq_u y == CONST\ unot\ (x =_u y)$   
 $x \notin_u A == CONST\ unot\ (CONST\ bop\ (\in)\ x\ A)$

**declare** *true-upred-def* [*upred-defs*]  
**declare** *false-upred-def* [*upred-defs*]  
**declare** *conj-upred-def* [*upred-defs*]  
**declare** *disj-upred-def* [*upred-defs*]  
**declare** *not-upred-def* [*upred-defs*]  
**declare** *diff-upred-def* [*upred-defs*]  
**declare** *subst-upd-uvar-def* [*upred-defs*]  
**declare** *cond-subst-def* [*upred-defs*]  
**declare** *par-subst-def* [*upred-defs*]  
**declare** *subst-del-def* [*upred-defs*]  
**declare** *unrest-usubst-def* [*upred-defs*]  
**declare** *uexpr-defs* [*upred-defs*]

**lemma** *true-alt-def*:  $true = \ll True \gg$   
**by** (*pred-auto*)

**lemma** *false-alt-def*:  $false = \ll False \gg$   
**by** (*pred-auto*)

**declare** *true-alt-def* [*THEN sym, simp*]  
**declare** *false-alt-def* [*THEN sym, simp*]

## 10.3 Unrestriction Laws

**lemma** *unrest-allE*:  
 $\ll \Sigma \# P; P = true \implies Q; P = false \implies Q \gg \implies Q$   
**by** (*pred-auto*)

**lemma** *unrest-true* [*unrest*]:  $x \# true$   
**by** (*pred-auto*)

**lemma** *unrest-false* [*unrest*]:  $x \# false$   
**by** (*pred-auto*)

**lemma** *unrest-conj* [*unrest*]:  $\ll x \# (P :: 'α\ upred); x \# Q \gg \implies x \# P \wedge Q$   
**by** (*pred-auto*)

**lemma** *unrest-disj* [*unrest*]:  $\ll x \# (P :: 'α\ upred); x \# Q \gg \implies x \# P \vee Q$   
**by** (*pred-auto*)

**lemma** *unrest-UNIF* [*unrest*]:  
 $\ll (\bigwedge i. x \# P(i)); (\bigwedge i. x \# Q(i)) \gg \implies x \# (\bigcap i \mid P(i) \cdot Q(i))$   
**by** (*pred-auto*)

**lemma** *unrest-USUP* [*unrest*]:  
 $\ll (\bigwedge i. x \# P(i)); (\bigwedge i. x \# Q(i)) \gg \implies x \# (\bigcup i \mid P(i) \cdot Q(i))$   
**by** (*pred-auto*)

**lemma** *unrest-UINF-mem* [*unrest*]:  
 $\llbracket (\bigwedge i. i \in A \implies x \# P(i)) \rrbracket \implies x \# (\bigcap i \in A \cdot P(i))$   
**by** (*pred-simp*, *metis*)

**lemma** *unrest-USUP-mem* [*unrest*]:  
 $\llbracket (\bigwedge i. i \in A \implies x \# P(i)) \rrbracket \implies x \# (\bigcup i \in A \cdot P(i))$   
**by** (*pred-simp*, *metis*)

**lemma** *unrest-impl* [*unrest*]:  $\llbracket x \# P; x \# Q \rrbracket \implies x \# P \Rightarrow Q$   
**by** (*pred-auto*)

**lemma** *unrest-iff* [*unrest*]:  $\llbracket x \# P; x \# Q \rrbracket \implies x \# P \Leftrightarrow Q$   
**by** (*pred-auto*)

**lemma** *unrest-not* [*unrest*]:  $x \# (P :: 'a \text{ upred}) \implies x \# (\neg P)$   
**by** (*pred-auto*)

The sublens proviso can be thought of as membership below.

**lemma** *unrest-ex-in* [*unrest*]:  
 $\llbracket \text{mwb-lens } y; x \subseteq_L y \rrbracket \implies x \# (\exists y \cdot P)$   
**by** (*pred-auto*)

**declare** *sublens-refl* [*simp*]  
**declare** *lens-plus-ub* [*simp*]  
**declare** *lens-plus-right-sublens* [*simp*]  
**declare** *comp-wb-lens* [*simp*]  
**declare** *comp-mwb-lens* [*simp*]  
**declare** *plus-mwb-lens* [*simp*]

**lemma** *unrest-ex-diff* [*unrest*]:  
**assumes**  $x \bowtie y$   $y \# P$   
**shows**  $y \# (\exists x \cdot P)$   
**using** *assms lens-indep-comm*  
**by** (*rel-simp'*, *fastforce*)

**lemma** *unrest-all-in* [*unrest*]:  
 $\llbracket \text{mwb-lens } y; x \subseteq_L y \rrbracket \implies x \# (\forall y \cdot P)$   
**by** (*pred-auto*)

**lemma** *unrest-all-diff* [*unrest*]:  
**assumes**  $x \bowtie y$   $y \# P$   
**shows**  $y \# (\forall x \cdot P)$   
**using** *assms*  
**by** (*pred-simp*, *simp-all add: lens-indep-comm*)

**lemma** *unrest-var-res-diff* [*unrest*]:  
**assumes**  $x \bowtie y$   
**shows**  $y \# (P \upharpoonright_v x)$   
**using** *assms* **by** (*pred-auto*)

**lemma** *unrest-var-res-in* [*unrest*]:  
**assumes**  $\text{mwb-lens } x y \subseteq_L x y \# P$   
**shows**  $y \# (P \upharpoonright_v x)$   
**using** *assms*



```

apply (pred-auto)
apply fastforce
apply (metis (no-types, lifting) mwb-lens-weak weak-lens.put-get)
done

```

```

lemma unrest-shEx [unrest]:
  assumes  $\bigwedge y. x \# P(y)$ 
  shows  $x \# (\exists y. P(y))$ 
  using assms by (pred-auto)

```

```

lemma unrest-shAll [unrest]:
  assumes  $\bigwedge y. x \# P(y)$ 
  shows  $x \# (\forall y. P(y))$ 
  using assms by (pred-auto)

```

```

lemma unrest-closure [unrest]:
   $x \# [P]_u$ 
  by (pred-auto)

```

## 10.4 Used-by laws

```

lemma usedBy-not [unrest]:
   $\llbracket x \Downarrow P \rrbracket \implies x \Downarrow (\neg P)$ 
  by (pred-simp)

```

```

lemma usedBy-conj [unrest]:
   $\llbracket x \Downarrow P; x \Downarrow Q \rrbracket \implies x \Downarrow (P \wedge Q)$ 
  by (pred-simp)

```

```

lemma usedBy-disj [unrest]:
   $\llbracket x \Downarrow P; x \Downarrow Q \rrbracket \implies x \Downarrow (P \vee Q)$ 
  by (pred-simp)

```

```

lemma usedBy-impl [unrest]:
   $\llbracket x \Downarrow P; x \Downarrow Q \rrbracket \implies x \Downarrow (P \Rightarrow Q)$ 
  by (pred-simp)

```

```

lemma usedBy-iff [unrest]:
   $\llbracket x \Downarrow P; x \Downarrow Q \rrbracket \implies x \Downarrow (P \Leftrightarrow Q)$ 
  by (pred-simp)

```

## 10.5 Substitution Laws

Substitution is monotone

```

lemma subst-mono:  $P \sqsubseteq Q \implies (\sigma \uparrow P) \sqsubseteq (\sigma \uparrow Q)$ 
  by (pred-auto)

```

```

lemma subst-true [usubst]:  $\sigma \uparrow \text{true} = \text{true}$ 
  by (pred-auto)

```

```

lemma subst-false [usubst]:  $\sigma \uparrow \text{false} = \text{false}$ 
  by (pred-auto)

```

```

lemma subst-not [usubst]:  $\sigma \uparrow (\neg P) = (\neg \sigma \uparrow P)$ 
  by (pred-auto)

```

**lemma** *subst-impl* [*usubst*]:  $\sigma \dagger (P \Rightarrow Q) = (\sigma \dagger P \Rightarrow \sigma \dagger Q)$   
**by** (*pred-auto*)

**lemma** *subst-iff* [*usubst*]:  $\sigma \dagger (P \Leftrightarrow Q) = (\sigma \dagger P \Leftrightarrow \sigma \dagger Q)$   
**by** (*pred-auto*)

**lemma** *subst-disj* [*usubst*]:  $\sigma \dagger (P \vee Q) = (\sigma \dagger P \vee \sigma \dagger Q)$   
**by** (*pred-auto*)

**lemma** *subst-conj* [*usubst*]:  $\sigma \dagger (P \wedge Q) = (\sigma \dagger P \wedge \sigma \dagger Q)$   
**by** (*pred-auto*)

**lemma** *subst-sup* [*usubst*]:  $\sigma \dagger (P \sqcap Q) = (\sigma \dagger P \sqcap \sigma \dagger Q)$   
**by** (*pred-auto*)

**lemma** *subst-inf* [*usubst*]:  $\sigma \dagger (P \sqcup Q) = (\sigma \dagger P \sqcup \sigma \dagger Q)$   
**by** (*pred-auto*)

**lemma** *subst-UINF* [*usubst*]:  $\sigma \dagger (\prod i \mid P(i) \cdot Q(i)) = (\prod i \mid (\sigma \dagger P(i)) \cdot (\sigma \dagger Q(i)))$   
**by** (*pred-auto*)

**lemma** *subst-USUP* [*usubst*]:  $\sigma \dagger (\bigsqcup i \mid P(i) \cdot Q(i)) = (\bigsqcup i \mid (\sigma \dagger P(i)) \cdot (\sigma \dagger Q(i)))$   
**by** (*pred-auto*)

**lemma** *subst-closure* [*usubst*]:  $\sigma \dagger [P]_u = [P]_u$   
**by** (*pred-auto*)

**lemma** *subst-shEx* [*usubst*]:  $\sigma \dagger (\exists x \cdot P(x)) = (\exists x \cdot \sigma \dagger P(x))$   
**by** (*pred-auto*)

**lemma** *subst-shAll* [*usubst*]:  $\sigma \dagger (\forall x \cdot P(x)) = (\forall x \cdot \sigma \dagger P(x))$   
**by** (*pred-auto*)

TODO: Generalise the quantifier substitution laws to n-ary substitutions

**lemma** *subst-ex-same* [*usubst*]:  
 $mwb\text{-}lens\ x \Longrightarrow \sigma(x \mapsto_s v) \dagger (\exists x \cdot P) = \sigma \dagger (\exists x \cdot P)$   
**by** (*pred-auto*)

**lemma** *subst-ex-same'* [*usubst*]:  
 $mwb\text{-}lens\ x \Longrightarrow \sigma(x \mapsto_s v) \dagger (\exists \&x \cdot P) = \sigma \dagger (\exists \&x \cdot P)$   
**by** (*pred-auto*)

**lemma** *subst-ex-indep* [*usubst*]:  
**assumes**  $x \bowtie y \ y \nmid v$   
**shows**  $(\exists y \cdot P) \llbracket v/x \rrbracket = (\exists y \cdot P \llbracket v/x \rrbracket)$   
**using** *assms*  
**apply** (*pred-auto*)  
**using** *lens-indep-comm* **apply** *fastforce* +  
**done**

**lemma** *subst-ex-unrest* [*usubst*]:  
 $x \nmid \sigma \Longrightarrow \sigma \dagger (\exists x \cdot P) = (\exists x \cdot \sigma \dagger P)$   
**by** (*pred-auto*)

**lemma** *subst-all-same* [usubst]:

*mw-lens*  $x \implies \sigma(x \mapsto_s v) \uparrow (\forall x \cdot P) = \sigma \uparrow (\forall x \cdot P)$   
**by** (*simp add: id-subst subst-unrest unrest-all-in*)

**lemma** *subst-all-indep* [usubst]:

**assumes**  $x \bowtie y \cdot y \nparallel v$   
**shows**  $(\forall y \cdot P) \llbracket v/x \rrbracket = (\forall y \cdot P \llbracket v/x \rrbracket)$   
**using** *assms*  
**by** (*pred-simp, simp-all add: lens-indep-comm*)

**lemma** *msubst-true* [usubst]:  $\text{true} \llbracket x \rightarrow v \rrbracket = \text{true}$

**by** (*pred-auto*)

**lemma** *msubst-false* [usubst]:  $\text{false} \llbracket x \rightarrow v \rrbracket = \text{false}$

**by** (*pred-auto*)

**lemma** *msubst-not* [usubst]:  $(\neg P(x)) \llbracket x \rightarrow v \rrbracket = (\neg ((P x) \llbracket x \rightarrow v \rrbracket))$

**by** (*pred-auto*)

**lemma** *msubst-not-2* [usubst]:  $(\neg P x y) \llbracket (x,y) \rightarrow v \rrbracket = (\neg ((P x y) \llbracket (x,y) \rightarrow v \rrbracket))$

**by** (*pred-auto*)<sup>+</sup>

**lemma** *msubst-disj* [usubst]:  $(P(x) \vee Q(x)) \llbracket x \rightarrow v \rrbracket = ((P(x)) \llbracket x \rightarrow v \rrbracket \vee (Q(x)) \llbracket x \rightarrow v \rrbracket)$

**by** (*pred-auto*)

**lemma** *msubst-disj-2* [usubst]:  $(P x y \vee Q x y) \llbracket (x,y) \rightarrow v \rrbracket = ((P x y) \llbracket (x,y) \rightarrow v \rrbracket \vee (Q x y) \llbracket (x,y) \rightarrow v \rrbracket)$

**by** (*pred-auto*)<sup>+</sup>

**lemma** *msubst-conj* [usubst]:  $(P(x) \wedge Q(x)) \llbracket x \rightarrow v \rrbracket = ((P(x)) \llbracket x \rightarrow v \rrbracket \wedge (Q(x)) \llbracket x \rightarrow v \rrbracket)$

**by** (*pred-auto*)

**lemma** *msubst-conj-2* [usubst]:  $(P x y \wedge Q x y) \llbracket (x,y) \rightarrow v \rrbracket = ((P x y) \llbracket (x,y) \rightarrow v \rrbracket \wedge (Q x y) \llbracket (x,y) \rightarrow v \rrbracket)$

**by** (*pred-auto*)<sup>+</sup>

**lemma** *msubst-implies* [usubst]:

$(P x \Rightarrow Q x) \llbracket x \rightarrow v \rrbracket = ((P x) \llbracket x \rightarrow v \rrbracket \Rightarrow (Q x) \llbracket x \rightarrow v \rrbracket)$

**by** (*pred-auto*)

**lemma** *msubst-implies-2* [usubst]:

$(P x y \Rightarrow Q x y) \llbracket (x,y) \rightarrow v \rrbracket = ((P x y) \llbracket (x,y) \rightarrow v \rrbracket \Rightarrow (Q x y) \llbracket (x,y) \rightarrow v \rrbracket)$

**by** (*pred-auto*)<sup>+</sup>

**lemma** *msubst-shAll* [usubst]:

$(\forall x \cdot P x y) \llbracket y \rightarrow v \rrbracket = (\forall x \cdot (P x y) \llbracket y \rightarrow v \rrbracket)$

**by** (*pred-auto*)

**lemma** *msubst-shAll-2* [usubst]:

$(\forall x \cdot P x y z) \llbracket (y,z) \rightarrow v \rrbracket = (\forall x \cdot (P x y z) \llbracket (y,z) \rightarrow v \rrbracket)$

**by** (*pred-auto*)<sup>+</sup>

## 10.6 Sandbox for conjectures

**definition** *utp-sandbox* ::  $'\alpha \text{ upred} \Rightarrow \text{bool}$  (*TRY*'(-')) **where**

*TRY*(*P*) = (*P* = *undefined*)

**translations**

$P \leq \text{CONST utp-sandbox } P$

end

## 11 Alphabet Manipulation

```
theory utp-alphabet
  imports
    utp-pred utp-usedby
begin
```

### 11.1 Preliminaries

Alphabets are simply types that characterise the state-space of an expression. Thus the Isabelle type system ensures that predicates cannot refer to variables not in the alphabet as this would be a type error. Often one would like to add or remove additional variables, for example if we wish to have a predicate which ranges only a smaller state-space, and then lift it into a predicate over a larger one. This is useful, for example, when dealing with relations which refer only to undashed variables (conditions) since we can use the type system to ensure well-formedness.

In this theory we will set up operators for extending and contracting an alphabet. We first set up a theorem attribute for alphabet laws and a tactic.

```
named-theorems alpha
```

```
method alpha-tac = (simp add: alpha unrest)?
```

### 11.2 Alphabet Extrusion

Alter an alphabet by application of a lens that demonstrates how the smaller alphabet ( $\beta$ ) injects into the larger alphabet ( $\alpha$ ). This changes the type of the expression so it is parametrised over the large alphabet. We do this by using the lens *get* function to extract the smaller state binding, and then apply this to the expression.

We call this "extrusion" rather than "extension" because if the extension lens is bijective then it does not extend the alphabet. Nevertheless, it does have an effect because the type will be different which can be useful when converting predicates with equivalent alphabets.

**lift-definition**  $aext :: ('a, 'b) uexpr \Rightarrow ('b, 'a) lens \Rightarrow ('a, 'a) uexpr$  (**infixr**  $\oplus_p$  95)  
**is**  $\lambda P x b. P (get_x b)$  .

```
update-uexpr-rep-eq-thms
```

Next we prove some of the key laws. Extending an alphabet twice is equivalent to extending by the composition of the two lenses.

**lemma** *aext-twice*:  $(P \oplus_p a) \oplus_p b = P \oplus_p (a ;_L b)$   
**by** (*pred-auto*)

The bijective  $\Sigma$  lens identifies the source and view types. Thus an alphabet extension using this has no effect.

**lemma** *aext-id* [*simp*]:  $P \oplus_p 1_L = P$   
**by** (*pred-auto*)

Literals do not depend on any variables, and thus applying an alphabet extension only alters the predicate's type, and not its valuation .

**lemma** *aext-lit* [*simp*]:  $\langle\langle v \rangle\rangle \oplus_p a = \langle\langle v \rangle\rangle$

by (pred-auto)

**lemma** aext-zero [simp]:  $0 \oplus_p a = 0$   
by (pred-auto)

**lemma** aext-one [simp]:  $1 \oplus_p a = 1$   
by (pred-auto)

**lemma** aext-numeral [simp]:  $\text{numeral } n \oplus_p a = \text{numeral } n$   
by (pred-auto)

**lemma** aext-true [simp]:  $\text{true} \oplus_p a = \text{true}$   
by (pred-auto)

**lemma** aext-false [simp]:  $\text{false} \oplus_p a = \text{false}$   
by (pred-auto)

**lemma** aext-not [alpha]:  $(\neg P) \oplus_p x = (\neg (P \oplus_p x))$   
by (pred-auto)

**lemma** aext-and [alpha]:  $(P \wedge Q) \oplus_p x = (P \oplus_p x \wedge Q \oplus_p x)$   
by (pred-auto)

**lemma** aext-or [alpha]:  $(P \vee Q) \oplus_p x = (P \oplus_p x \vee Q \oplus_p x)$   
by (pred-auto)

**lemma** aext-imp [alpha]:  $(P \Rightarrow Q) \oplus_p x = (P \oplus_p x \Rightarrow Q \oplus_p x)$   
by (pred-auto)

**lemma** aext-iff [alpha]:  $(P \Leftrightarrow Q) \oplus_p x = (P \oplus_p x \Leftrightarrow Q \oplus_p x)$   
by (pred-auto)

**lemma** aext-shAll [alpha]:  $(\forall x \cdot P(x)) \oplus_p a = (\forall x \cdot P(x) \oplus_p a)$   
by (pred-auto)

**lemma** aext-UINF-ind [alpha]:  $(\bigcap x \cdot P x) \oplus_p a = (\bigcap x \cdot (P x \oplus_p a))$   
by (pred-auto)

**lemma** aext-UINF-mem [alpha]:  $(\bigcap x \in A \cdot P x) \oplus_p a = (\bigcap x \in A \cdot (P x \oplus_p a))$   
by (pred-auto)

Alphabet extension distributes through the function liftings.

**lemma** aext-uop [alpha]:  $\text{uop } f \ u \oplus_p a = \text{uop } f \ (u \oplus_p a)$   
by (pred-auto)

**lemma** aext-bop [alpha]:  $\text{bop } f \ u \ v \oplus_p a = \text{bop } f \ (u \oplus_p a) \ (v \oplus_p a)$   
by (pred-auto)

**lemma** aext-trop [alpha]:  $\text{trop } f \ u \ v \ w \oplus_p a = \text{trop } f \ (u \oplus_p a) \ (v \oplus_p a) \ (w \oplus_p a)$   
by (pred-auto)

**lemma** aext-qtrop [alpha]:  $\text{qtrop } f \ u \ v \ w \ x \oplus_p a = \text{qtrop } f \ (u \oplus_p a) \ (v \oplus_p a) \ (w \oplus_p a) \ (x \oplus_p a)$   
by (pred-auto)

**lemma** aext-plus [alpha]:

$(x + y) \oplus_p a = (x \oplus_p a) + (y \oplus_p a)$   
**by** (*pred-auto*)

**lemma** *aext-minus* [*alpha*]:  
 $(x - y) \oplus_p a = (x \oplus_p a) - (y \oplus_p a)$   
**by** (*pred-auto*)

**lemma** *aext-uminus* [*simp*]:  
 $(-x) \oplus_p a = -(x \oplus_p a)$   
**by** (*pred-auto*)

**lemma** *aext-times* [*alpha*]:  
 $(x * y) \oplus_p a = (x \oplus_p a) * (y \oplus_p a)$   
**by** (*pred-auto*)

**lemma** *aext-divide* [*alpha*]:  
 $(x / y) \oplus_p a = (x \oplus_p a) / (y \oplus_p a)$   
**by** (*pred-auto*)

Extending a variable expression over  $x$  is equivalent to composing  $x$  with the alphabet, thus effectively yielding a variable whose source is the large alphabet.

**lemma** *aext-var* [*alpha*]:  
 $\text{var } x \oplus_p a = \text{var } (x ;_L a)$   
**by** (*pred-auto*)

**lemma** *aext-ulambda* [*alpha*]:  $((\lambda x \cdot P(x)) \oplus_p a) = (\lambda x \cdot P(x) \oplus_p a)$   
**by** (*pred-auto*)

Alphabet extension is monotonic and continuous.

**lemma** *aext-mono*:  $P \sqsubseteq Q \implies P \oplus_p a \sqsubseteq Q \oplus_p a$   
**by** (*pred-auto*)

**lemma** *aext-cont* [*alpha*]:  $\text{vwb-lens } a \implies (\bigsqcap A) \oplus_p a = (\bigsqcap P \in A. P \oplus_p a)$   
**by** (*pred-simp*)

If a variable is unrestricted in a predicate, then the extended variable is unrestricted in the predicate with an alphabet extension.

**lemma** *unrest-aext* [*unrest*]:  
 $\llbracket \text{vwb-lens } a; x \# p \rrbracket \implies \text{unrest } (x ;_L a) (p \oplus_p a)$   
**by** (*transfer, simp add: lens-comp-def*)

If a given variable (or alphabet)  $b$  is independent of the extension lens  $a$ , that is, it is outside the original state-space of  $p$ , then it follows that once  $p$  is extended by  $a$  then  $b$  cannot be restricted.

**lemma** *unrest-aext-indep* [*unrest*]:  
 $a \bowtie b \implies b \# (p \oplus_p a)$   
**by** *pred-auto*

### 11.3 Expression Alphabet Restriction

Restrict an alphabet by application of a lens that demonstrates how the smaller alphabet ( $\beta$ ) injects into the larger alphabet ( $\alpha$ ). Unlike extension, this operation can lose information if the expressions refers to variables in the larger alphabet.

**lift-definition** *arestr* ::  $('a, 'a) \text{ uexpr} \Rightarrow ('b, 'a) \text{ lens} \Rightarrow ('a, 'b) \text{ uexpr}$  (**infixr**  $\vdash_e$  90)

is  $\lambda P x b. P \text{ (create}_x b) \text{ .}$

**update-uexpr-rep-eq-thms**

**lemma** *arestr-id* [simp]:  $P \upharpoonright_e 1_L = P$   
 by (*pred-auto*)

**lemma** *arestr-aext* [simp]:  $\text{mwb-lens } a \implies (P \oplus_p a) \upharpoonright_e a = P$   
 by (*pred-auto*)

If an expression's alphabet can be divided into two disjoint sections and the expression does not depend on the second half then restricting the expression to the first half is loss-less.

**lemma** *aext-arestr* [*alpha*]:  
 assumes  $\text{mwb-lens } a \text{ bij-lens } (a +_L b) \text{ } a \bowtie b \text{ } b \# P$   
 shows  $(P \upharpoonright_e a) \oplus_p a = P$   
**proof** –  
 from *assms*(2) have  $1_L \subseteq_L a +_L b$   
 by (*simp add: bij-lens-equiv-id lens-equiv-def*)  
 with *assms*(1,3,4) **show** ?thesis  
 apply (*auto simp add: id-lens-def lens-plus-def sublens-def lens-comp-def prod.case-eq-if*)  
 apply (*pred-simp*)  
 apply (*metis lens-indep-comm mwb-lens-weak weak-lens.put-get*)  
 done  
**qed**

Alternative formulation of the above law using *used-by* instead of *unrestriction*.

**lemma** *aext-arestr'* [*alpha*]:  
 assumes  $a \# P$   
 shows  $(P \upharpoonright_e a) \oplus_p a = P$   
 by (*rel-simp, metis assms lens-override-def usedBy-uexpr.rep-eq*)

**lemma** *arestr-lit* [simp]:  $\langle\langle v \rangle\rangle \upharpoonright_e a = \langle\langle v \rangle\rangle$   
 by (*pred-auto*)

**lemma** *arestr-zero* [simp]:  $0 \upharpoonright_e a = 0$   
 by (*pred-auto*)

**lemma** *arestr-one* [simp]:  $1 \upharpoonright_e a = 1$   
 by (*pred-auto*)

**lemma** *arestr-numeral* [simp]:  $\text{numeral } n \upharpoonright_e a = \text{numeral } n$   
 by (*pred-auto*)

**lemma** *arestr-var* [*alpha*]:  
 $\text{var } x \upharpoonright_e a = \text{var } (x /_L a)$   
 by (*pred-auto*)

**lemma** *arestr-true* [simp]:  $\text{true} \upharpoonright_e a = \text{true}$   
 by (*pred-auto*)

**lemma** *arestr-false* [simp]:  $\text{false} \upharpoonright_e a = \text{false}$   
 by (*pred-auto*)

**lemma** *arestr-not* [*alpha*]:  $(\neg P) \upharpoonright_e a = (\neg (P \upharpoonright_e a))$   
 by (*pred-auto*)

**lemma** *arestr-and* [*alpha*]:  $(P \wedge Q) \upharpoonright_e x = (P \upharpoonright_e x \wedge Q \upharpoonright_e x)$   
**by** (*pred-auto*)

**lemma** *arestr-or* [*alpha*]:  $(P \vee Q) \upharpoonright_e x = (P \upharpoonright_e x \vee Q \upharpoonright_e x)$   
**by** (*pred-auto*)

**lemma** *arestr-imp* [*alpha*]:  $(P \Rightarrow Q) \upharpoonright_e x = (P \upharpoonright_e x \Rightarrow Q \upharpoonright_e x)$   
**by** (*pred-auto*)

## 11.4 Predicate Alphabet Restriction

In order to restrict the variables of a predicate, we also need to existentially quantify away the other variables. We can't do this at the level of expressions, as quantifiers are not applicable here. Consequently, we need a specialised version of alphabet restriction for predicates. It both restricts the variables using quantification and then removes them from the alphabet type using expression restriction.

**definition** *upred-ares* ::  $'\alpha \text{ upred} \Rightarrow (' \beta \Longrightarrow ' \alpha) \Rightarrow ' \beta \text{ upred}$   
**where** [*upred-defs*]:  $\text{upred-ares } P \ a = (P \upharpoonright_v a) \upharpoonright_e a$

**syntax**

*-upred-ares* ::  $\text{logic} \Rightarrow \text{salpha} \Rightarrow \text{logic}$  (**infixl**  $\upharpoonright_p$  90)

**translations**

*-upred-ares*  $P \ a == \text{CONST upred-ares } P \ a$

**lemma** *upred-aext-ares* [*alpha*]:  
 $\text{vwb-lens } a \Longrightarrow P \oplus_p a \upharpoonright_p a = P$   
**by** (*pred-auto*)

**lemma** *upred-ares-aext* [*alpha*]:  
 $a \Vdash P \Longrightarrow (P \upharpoonright_p a) \oplus_p a = P$   
**by** (*pred-auto*)

**lemma** *upred-arestr-lit* [*simp*]:  $\ll v \gg \upharpoonright_p a = \ll v \gg$   
**by** (*pred-auto*)

**lemma** *upred-arestr-true* [*simp*]:  $\text{true} \upharpoonright_p a = \text{true}$   
**by** (*pred-auto*)

**lemma** *upred-arestr-false* [*simp*]:  $\text{false} \upharpoonright_p a = \text{false}$   
**by** (*pred-auto*)

**lemma** *upred-arestr-or* [*alpha*]:  $(P \vee Q) \upharpoonright_p x = (P \upharpoonright_p x \vee Q \upharpoonright_p x)$   
**by** (*pred-auto*)

## 11.5 Alphabet Lens Laws

**lemma** *alpha-in-var* [*alpha*]:  $x ;_L \text{fst}_L = \text{in-var } x$   
**by** (*simp add: in-var-def*)

**lemma** *alpha-out-var* [*alpha*]:  $x ;_L \text{snd}_L = \text{out-var } x$   
**by** (*simp add: out-var-def*)

**lemma** *in-var-prod-lens* [*alpha*]:



*wb-lens*  $Y \implies \text{in-var } x ;_L (X \times_L Y) = \text{in-var } (x ;_L X)$   
**by** (*simp add: in-var-def prod-as-plus lens-comp-assoc fst-lens-plus*)

**lemma** *out-var-prod-lens* [*alpha*]:  
*wb-lens*  $X \implies \text{out-var } x ;_L (X \times_L Y) = \text{out-var } (x ;_L Y)$   
**apply** (*simp add: out-var-def prod-as-plus lens-comp-assoc*)  
**apply** (*subst snd-lens-plus*)  
**using** *comp-wb-lens fst-vwb-lens vwb-lens-wb* **apply** *blast*  
**apply** (*simp add: alpha-in-var alpha-out-var*)  
**apply** (*simp*)  
**done**

## 11.6 Substitution Alphabet Extension

This allows us to extend the alphabet of a substitution, in a similar way to expressions.

**definition** *subst-ext* ::  $'\alpha \text{ usubst} \Rightarrow ('\alpha \implies '\beta) \Rightarrow '\beta \text{ usubst}$  (**infix**  $\oplus_s$  65) **where**  
*[upred-defs]*:  $\sigma \oplus_s x = (\lambda s. \text{put}_x s (\sigma (\text{get}_x s)))$

**lemma** *id-subst-ext* [*usubst*]:  
*wb-lens*  $x \implies \text{id} \oplus_s x = \text{id}$   
**by** *pred-auto*

**lemma** *upd-subst-ext* [*alpha*]:  
*vwb-lens*  $x \implies \sigma(y \mapsto_s v) \oplus_s x = (\sigma \oplus_s x)(\&x:y \mapsto_s v \oplus_p x)$   
**by** *pred-auto*

**lemma** *apply-subst-ext* [*alpha*]:  
*vwb-lens*  $x \implies (\sigma \dagger e) \oplus_p x = (\sigma \oplus_s x) \dagger (e \oplus_p x)$   
**by** (*pred-auto*)

**lemma** *aext-upred-eq* [*alpha*]:  
 $((e =_u f) \oplus_p a) = ((e \oplus_p a) =_u (f \oplus_p a))$   
**by** (*pred-auto*)

**lemma** *subst-aext-comp* [*usubst*]:  
*vwb-lens*  $a \implies (\sigma \oplus_s a) \circ (\varrho \oplus_s a) = (\sigma \circ \varrho) \oplus_s a$   
**by** *pred-auto*

## 11.7 Substitution Alphabet Restriction

This allows us to reduce the alphabet of a substitution, in a similar way to expressions.

**definition** *subst-res* ::  $'\alpha \text{ usubst} \Rightarrow (' \beta \implies ' \alpha) \Rightarrow ' \beta \text{ usubst}$  (**infix**  $\upharpoonright_s$  65) **where**  
*[upred-defs]*:  $\sigma \upharpoonright_s x = (\lambda s. \text{get}_x (\sigma (\text{create}_x s)))$

**lemma** *id-subst-res* [*usubst*]:  
*mwb-lens*  $x \implies \text{id} \upharpoonright_s x = \text{id}$   
**by** *pred-auto*

**lemma** *upd-subst-res* [*alpha*]:  
*mwb-lens*  $x \implies \sigma(\&x:y \mapsto_s v) \upharpoonright_s x = (\sigma \upharpoonright_s x)(\&y \mapsto_s v \upharpoonright_e x)$   
**by** (*pred-auto*)

**lemma** *subst-ext-res* [*usubst*]:  
*mwb-lens*  $x \implies (\sigma \oplus_s x) \upharpoonright_s x = \sigma$

by (*pred-auto*)

**lemma** *unrest-subst-alpha-ext* [*unrest*]:  
 $x \bowtie y \implies x \sharp (P \oplus_s y)$   
 by (*pred-simp robust, metis lens-indep-def*)  
 end

## 12 Lifting Expressions

**theory** *utp-lift*  
**imports**  
*utp-alphabet*  
**begin**

### 12.1 Lifting definitions

We define operators for converting an expression to and from a relational state space with the help of alphabet extrusion and restriction. In general throughout Isabelle/UTP we adopt the notation  $\lceil P \rceil$  with some subscript to denote lifting an expression into a larger alphabet, and  $\lfloor P \rfloor$  for dropping into a smaller alphabet.

The following two functions lift and drop an expression, respectively, whose alphabet is  $'\alpha$ , into a product alphabet  $'\alpha \times '\beta$ . This allows us to deal with expressions which refer only to undashed variables, and use the type-system to ensure this.

**abbreviation** *lift-pre* ::  $('a, '\alpha) \text{ uexpr} \Rightarrow ('a, '\alpha \times '\beta) \text{ uexpr}$  ( $\lceil \cdot \rceil_{<}$ )  
**where**  $\lceil P \rceil_{<} \equiv P \oplus_p \text{fst}_L$

**abbreviation** *drop-pre* ::  $('a, '\alpha \times '\beta) \text{ uexpr} \Rightarrow ('a, '\alpha) \text{ uexpr}$  ( $\lfloor \cdot \rfloor_{<}$ )  
**where**  $\lfloor P \rfloor_{<} \equiv P \upharpoonright_e \text{fst}_L$

The following two functions lift and drop an expression, respectively, whose alphabet is  $'\beta$ , into a product alphabet  $'\alpha \times '\beta$ . This allows us to deal with expressions which refer only to dashed variables.

**abbreviation** *lift-post* ::  $('a, '\beta) \text{ uexpr} \Rightarrow ('a, '\alpha \times '\beta) \text{ uexpr}$  ( $\lceil \cdot \rceil_{>}$ )  
**where**  $\lceil P \rceil_{>} \equiv P \oplus_p \text{snd}_L$

**abbreviation** *drop-post* ::  $('a, '\alpha \times '\beta) \text{ uexpr} \Rightarrow ('a, '\beta) \text{ uexpr}$  ( $\lfloor \cdot \rfloor_{>}$ )  
**where**  $\lfloor P \rfloor_{>} \equiv P \upharpoonright_e \text{snd}_L$

### 12.2 Lifting Laws

With the help of our alphabet laws, we can prove some intuitive laws about alphabet lifting. For example, lifting variables yields an unprimed or primed relational variable expression, respectively.

**lemma** *lift-pre-var* [*simp*]:  
 $\lceil \text{var } x \rceil_{<} = \$x$   
 by (*alpha-tac*)

**lemma** *lift-post-var* [*simp*]:  
 $\lceil \text{var } x \rceil_{>} = \$x'$   
 by (*alpha-tac*)

### 12.3 Substitution Laws

**lemma** *pre-var-subst* [*usubst*]:  
 $\sigma(\$x \mapsto_s \ll v \gg) \uparrow \lceil P \rceil_< = \sigma \uparrow \lceil P[\ll v \gg / \&x] \rceil_<$   
 by (*pred-simp*)

### 12.4 Unrestriction laws

Crucially, the lifting operators allow us to demonstrate unrestricted properties. For example, we can show that no primed variable is restricted in an expression over only the first element of the state-space product type.

**lemma** *unrest-dash-var-pre* [*unrest*]:  
 fixes  $x :: ('a \Rightarrow 'a)$   
 shows  $\$x' \# \lceil p \rceil_<$   
 by (*pred-auto*)

end

## 13 Predicate Calculus Laws

**theory** *utp-pred-laws*  
 imports *utp-pred*  
 begin

### 13.1 Propositional Logic

Showing that predicates form a Boolean Algebra (under the predicate operators as opposed to the lattice operators) gives us many useful laws.

**interpretation** *boolean-algebra* *diff-upred not-upred conj-upred* ( $\leq$ ) ( $<$ )  
*disj-upred false-upred true-upred*  
 by (*unfold-locales*; *pred-auto*)

**lemma** *taut-true* [*simp*]:  $\text{'true'}$   
 by (*pred-auto*)

**lemma** *taut-false* [*simp*]:  $\text{'false'}$  = *False*  
 by (*pred-auto*)

**lemma** *taut-conj*:  $\text{'A' } \wedge \text{'B'}$  =  $(\text{'A' } \wedge \text{'B'})$   
 by (*rel-auto*)

**lemma** *taut-conj-elim* [*elim!*]:  
 $\llbracket \text{'A' } \wedge \text{'B'}; \llbracket \text{'A'}; \text{'B' } \rrbracket \Rightarrow P \rrbracket \Rightarrow P$   
 by (*rel-auto*)

**lemma** *taut-refine-impl*:  $\llbracket Q \sqsubseteq P; \text{'P' } \rrbracket \Rightarrow \text{'Q'}$   
 by (*rel-auto*)

**lemma** *taut-shEx-elim*:  
 $\llbracket \text{'(} \exists x. P x \text{'}; \bigwedge x. \Sigma \# P x; \bigwedge x. \text{'P' } x \Rightarrow Q \rrbracket \Rightarrow Q$   
 by (*rel-blast*)

Linking refinement and HOL implication

**lemma** *refine-prop-intro*:

**assumes**  $\Sigma \# P \Sigma \# Q \text{ 'Q' } \Longrightarrow \text{ 'P' }$   
**shows**  $P \sqsubseteq Q$   
**using** *assms*  
**by** (*pred-auto*)

**lemma** *taut-not*:  $\Sigma \# P \Longrightarrow (\neg \text{ 'P' }) = \text{ '}\neg P\text{'}$   
**by** (*rel-auto*)

**lemma** *taut-shAll-intro*:  
 $\forall x. \text{ 'P x' } \Longrightarrow \forall x. x \cdot P x$   
**by** (*rel-auto*)

**lemma** *taut-shAll-intro-2*:  
 $\forall x y. \text{ 'P x y' } \Longrightarrow \forall (x, y). P x y$   
**by** (*rel-auto*)

**lemma** *taut-impl-intro*:  
 $\llbracket \Sigma \# P; \text{ 'P' } \Longrightarrow \text{ 'Q' } \rrbracket \Longrightarrow \text{ 'P } \Rightarrow Q\text{'}$   
**by** (*rel-auto*)

**lemma** *upred-eval-taut*:  
 $\text{ 'P } \llbracket \ll b \gg / \& \mathbf{v} \rrbracket \text{ ' } = \llbracket P \rrbracket_e b$   
**by** (*pred-auto*)

**lemma** *refBy-order*:  $P \sqsubseteq Q = \text{ 'Q } \Rightarrow P\text{'}$   
**by** (*pred-auto*)

**lemma** *conj-idem* [*simp*]:  $((P::'\alpha \text{ upred}) \wedge P) = P$   
**by** (*pred-auto*)

**lemma** *disj-idem* [*simp*]:  $((P::'\alpha \text{ upred}) \vee P) = P$   
**by** (*pred-auto*)

**lemma** *conj-comm*:  $((P::'\alpha \text{ upred}) \wedge Q) = (Q \wedge P)$   
**by** (*pred-auto*)

**lemma** *disj-comm*:  $((P::'\alpha \text{ upred}) \vee Q) = (Q \vee P)$   
**by** (*pred-auto*)

**lemma** *conj-subst*:  $P = R \Longrightarrow ((P::'\alpha \text{ upred}) \wedge Q) = (R \wedge Q)$   
**by** (*pred-auto*)

**lemma** *disj-subst*:  $P = R \Longrightarrow ((P::'\alpha \text{ upred}) \vee Q) = (R \vee Q)$   
**by** (*pred-auto*)

**lemma** *conj-assoc*:  $((P::'\alpha \text{ upred}) \wedge Q) \wedge S = (P \wedge (Q \wedge S))$   
**by** (*pred-auto*)

**lemma** *disj-assoc*:  $((P::'\alpha \text{ upred}) \vee Q) \vee S = (P \vee (Q \vee S))$   
**by** (*pred-auto*)

**lemma** *conj-disj-abs*:  $((P::'\alpha \text{ upred}) \wedge (P \vee Q)) = P$   
**by** (*pred-auto*)

**lemma** *disj-conj-abs*:  $((P::'\alpha \text{ upred}) \vee (P \wedge Q)) = P$

by (pred-auto)

**lemma** conj-disj-distr:  $((P::'\alpha \text{ upred}) \wedge (Q \vee R)) = ((P \wedge Q) \vee (P \wedge R))$   
 by (pred-auto)

**lemma** disj-conj-distr:  $((P::'\alpha \text{ upred}) \vee (Q \wedge R)) = ((P \vee Q) \wedge (P \vee R))$   
 by (pred-auto)

**lemma** true-disj-zero [simp]:  
 $(P \vee \text{true}) = \text{true} \quad (\text{true} \vee P) = \text{true}$   
 by (pred-auto)+

**lemma** true-conj-zero [simp]:  
 $(P \wedge \text{false}) = \text{false} \quad (\text{false} \wedge P) = \text{false}$   
 by (pred-auto)+

**lemma** false-sup [simp]:  $\text{false} \sqcap P = P \sqcap \text{false} = P$   
 by (pred-auto)+

**lemma** true-inf [simp]:  $\text{true} \sqcup P = P \sqcup \text{true} = P$   
 by (pred-auto)+

**lemma** imp-vacuous [simp]:  $(\text{false} \Rightarrow u) = \text{true}$   
 by (pred-auto)

**lemma** imp-true [simp]:  $(p \Rightarrow \text{true}) = \text{true}$   
 by (pred-auto)

**lemma** true-imp [simp]:  $(\text{true} \Rightarrow p) = p$   
 by (pred-auto)

**lemma** impl-mp1 [simp]:  $(P \wedge (P \Rightarrow Q)) = (P \wedge Q)$   
 by (pred-auto)

**lemma** impl-mp2 [simp]:  $((P \Rightarrow Q) \wedge P) = (Q \wedge P)$   
 by (pred-auto)

**lemma** impl-adjoin:  $((P \Rightarrow Q) \wedge R) = ((P \wedge R \Rightarrow Q \wedge R) \wedge R)$   
 by (pred-auto)

**lemma** impl-refine-intro:  
 $\llbracket Q_1 \sqsubseteq P_1; P_2 \sqsubseteq (P_1 \wedge Q_2) \rrbracket \Longrightarrow (P_1 \Rightarrow P_2) \sqsubseteq (Q_1 \Rightarrow Q_2)$   
 by (pred-auto)

**lemma** spec-refine:  
 $Q \sqsubseteq (P \wedge R) \Longrightarrow (P \Rightarrow Q) \sqsubseteq R$   
 by (rel-auto)

**lemma** impl-disjI:  $\llbracket 'P \Rightarrow R'; 'Q \Rightarrow R' \rrbracket \Longrightarrow '(P \vee Q) \Rightarrow R'$   
 by (rel-auto)

**lemma** conditional-iff:  
 $(P \Rightarrow Q) = (P \Rightarrow R) \longleftrightarrow 'P \Rightarrow (Q \Leftrightarrow R)'$   
 by (pred-auto)

**lemma** *p-and-not-p* [simp]:  $(P \wedge \neg P) = \text{false}$   
**by** (*pred-auto*)

**lemma** *p-or-not-p* [simp]:  $(P \vee \neg P) = \text{true}$   
**by** (*pred-auto*)

**lemma** *p-imp-p* [simp]:  $(P \Rightarrow P) = \text{true}$   
**by** (*pred-auto*)

**lemma** *p-iff-p* [simp]:  $(P \Leftrightarrow P) = \text{true}$   
**by** (*pred-auto*)

**lemma** *p-imp-false* [simp]:  $(P \Rightarrow \text{false}) = (\neg P)$   
**by** (*pred-auto*)

**lemma** *not-conj-deMorgans* [simp]:  $(\neg ((P::'\alpha \text{ upred}) \wedge Q)) = ((\neg P) \vee (\neg Q))$   
**by** (*pred-auto*)

**lemma** *not-disj-deMorgans* [simp]:  $(\neg ((P::'\alpha \text{ upred}) \vee Q)) = ((\neg P) \wedge (\neg Q))$   
**by** (*pred-auto*)

**lemma** *conj-disj-not-abs* [simp]:  $((P::'\alpha \text{ upred}) \wedge ((\neg P) \vee Q)) = (P \wedge Q)$   
**by** (*pred-auto*)

**lemma** *subsumption1*:  
 $'P \Rightarrow Q' \Longrightarrow (P \vee Q) = Q$   
**by** (*pred-auto*)

**lemma** *subsumption2*:  
 $'Q \Rightarrow P' \Longrightarrow (P \vee Q) = P$   
**by** (*pred-auto*)

**lemma** *neg-conj-cancel1*:  $(\neg P \wedge (P \vee Q)) = (\neg P \wedge Q :: '\alpha \text{ upred})$   
**by** (*pred-auto*)

**lemma** *neg-conj-cancel2*:  $(\neg Q \wedge (P \vee Q)) = (\neg Q \wedge P :: '\alpha \text{ upred})$   
**by** (*pred-auto*)

**lemma** *double-negation* [simp]:  $(\neg \neg (P::'\alpha \text{ upred})) = P$   
**by** (*pred-auto*)

**lemma** *true-not-false* [simp]:  $\text{true} \neq \text{false} \text{ false} \neq \text{true}$   
**by** (*pred-auto*)

**lemma** *closure-conj-distr*:  $([P]_u \wedge [Q]_u) = [P \wedge Q]_u$   
**by** (*pred-auto*)

**lemma** *closure-imp-distr*:  $'[P \Rightarrow Q]_u \Rightarrow [P]_u \Rightarrow [Q]_u'$   
**by** (*pred-auto*)

**lemma** *true-iff* [simp]:  $(P \Leftrightarrow \text{true}) = P$   
**by** (*pred-auto*)

**lemma** *taut-iff-eq*:  
 $'P \Leftrightarrow Q' \longleftrightarrow (P = Q)$

by (pred-auto)

**lemma** *impl-alt-def*:  $(P \Rightarrow Q) = (\neg P \vee Q)$   
by (pred-auto)

## 13.2 Lattice laws

**lemma** *uinf-or*:  
fixes  $P Q :: 'a \text{ upred}$   
shows  $(P \sqcap Q) = (P \vee Q)$   
by (pred-auto)

**lemma** *usup-and*:  
fixes  $P Q :: 'a \text{ upred}$   
shows  $(P \sqcup Q) = (P \wedge Q)$   
by (pred-auto)

**lemma** *UINF-alt-def*:  
 $(\bigsqcap i \mid A(i) \cdot P(i)) = (\bigsqcap i \cdot A(i) \wedge P(i))$   
by (rel-auto)

**lemma** *USUP-true* [simp]:  $(\bigsqcup P \mid F(P) \cdot \text{true}) = \text{true}$   
by (pred-auto)

**lemma** *UINF-mem-UNIV* [simp]:  $(\bigsqcap x \in \text{UNIV} \cdot P(x)) = (\bigsqcap x \cdot P(x))$   
by (pred-auto)

**lemma** *USUP-mem-UNIV* [simp]:  $(\bigsqcup x \in \text{UNIV} \cdot P(x)) = (\bigsqcup x \cdot P(x))$   
by (pred-auto)

**lemma** *USUP-false* [simp]:  $(\bigsqcup i \cdot \text{false}) = \text{false}$   
by (pred-simp)

**lemma** *USUP-mem-false* [simp]:  $I \neq \{\} \implies (\bigsqcup i \in I \cdot \text{false}) = \text{false}$   
by (rel-simp)

**lemma** *USUP-where-false* [simp]:  $(\bigsqcup i \mid \text{false} \cdot P(i)) = \text{true}$   
by (rel-auto)

**lemma** *UINF-true* [simp]:  $(\bigsqcap i \cdot \text{true}) = \text{true}$   
by (pred-simp)

**lemma** *UINF-ind-const* [simp]:  
 $(\bigsqcap i \cdot P) = P$   
by (rel-auto)

**lemma** *UINF-mem-true* [simp]:  $A \neq \{\} \implies (\bigsqcap i \in A \cdot \text{true}) = \text{true}$   
by (pred-auto)

**lemma** *UINF-false* [simp]:  $(\bigsqcap i \mid P(i) \cdot \text{false}) = \text{false}$   
by (pred-auto)

**lemma** *UINF-where-false* [simp]:  $(\bigsqcap i \mid \text{false} \cdot P(i)) = \text{false}$   
by (rel-auto)

**lemma** *UINF-cong-eq*:

$$\llbracket \bigwedge x. P_1(x) = P_2(x); \bigwedge x. 'P_1(x) \Rightarrow Q_1(x) =_u Q_2(x)' \rrbracket \Longrightarrow$$

$$(\bigcap x \mid P_1(x) \cdot Q_1(x)) = (\bigcap x \mid P_2(x) \cdot Q_2(x))$$
 by (unfold UINF-def, pred-simp, metis)

**lemma** UINF-as-Sup:  $(\bigcap P \in \mathcal{P} \cdot P) = \bigcap \mathcal{P}$   
 apply (simp add: upred-defs bop.rep-eq lit.rep-eq Sup-uexpr-def)  
 apply (pred-simp)  
 apply (rule cong[of Sup])  
 apply (auto)  
 done

**lemma** UINF-as-Sup-collect:  $(\bigcap P \in A \cdot f(P)) = (\bigcap P \in A. f(P))$   
 apply (simp add: upred-defs bop.rep-eq lit.rep-eq Sup-uexpr-def)  
 apply (pred-simp)  
 apply (simp add: Setcompr-eq-image)  
 done

**lemma** UINF-as-Sup-collect':  $(\bigcap P \cdot f(P)) = (\bigcap P. f(P))$   
 apply (simp add: upred-defs bop.rep-eq lit.rep-eq Sup-uexpr-def)  
 apply (pred-simp)  
 apply (simp add: full-SetCompr-eq)  
 done

**lemma** UINF-as-Sup-image:  $(\bigcap P \mid \ll P \gg \in_u \ll A \gg \cdot f(P)) = \bigcap (f \text{ ' } A)$   
 apply (simp add: upred-defs bop.rep-eq lit.rep-eq Sup-uexpr-def)  
 apply (pred-simp)  
 apply (rule cong[of Sup])  
 apply (auto)  
 done

**lemma** USUP-as-Inf:  $(\bigcup P \in \mathcal{P} \cdot P) = \bigcup \mathcal{P}$   
 apply (simp add: upred-defs bop.rep-eq lit.rep-eq Inf-uexpr-def)  
 apply (pred-simp)  
 apply (rule cong[of Inf])  
 apply (auto)  
 done

**lemma** USUP-as-Inf-collect:  $(\bigcup P \in A \cdot f(P)) = (\bigcup P \in A. f(P))$   
 apply (pred-simp)  
 apply (simp add: Setcompr-eq-image)  
 done

**lemma** USUP-as-Inf-collect':  $(\bigcup P \cdot f(P)) = (\bigcup P. f(P))$   
 apply (simp add: upred-defs bop.rep-eq lit.rep-eq Sup-uexpr-def)  
 apply (pred-simp)  
 apply (simp add: full-SetCompr-eq)  
 done

**lemma** USUP-as-Inf-image:  $(\bigcup P \in \mathcal{P} \cdot f(P)) = \bigcup (f \text{ ' } \mathcal{P})$   
 apply (simp add: upred-defs bop.rep-eq lit.rep-eq Inf-uexpr-def)  
 apply (pred-simp)  
 apply (rule cong[of Inf])  
 apply (auto)  
 done



**lemma** *USUP-image-eq* [simp]:  $USUP (\lambda i. \llbracket i \rrbracket \in_u \llbracket f \cdot A \rrbracket) g = (\bigsqcup_{i \in A} \cdot g(f(i)))$   
 by (pred-simp, rule-tac cong[of Inf Inf], auto)

**lemma** *UINF-image-eq* [simp]:  $UINF (\lambda i. \llbracket i \rrbracket \in_u \llbracket f \cdot A \rrbracket) g = (\bigsqcap_{i \in A} \cdot g(f(i)))$   
 by (pred-simp, rule-tac cong[of Sup Sup], auto)

**lemma** *subst-continuous* [usubst]:  $\sigma \uparrow (\bigsqcap A) = (\bigsqcap \{\sigma \uparrow P \mid P. P \in A\})$   
 by (simp add: UINF-as-Sup[THEN sym] usubst setcompr-eq-image)

**lemma** *not-UINF*:  $(\neg (\bigsqcap_{i \in A} \cdot P(i))) = (\bigsqcup_{i \in A} \cdot \neg P(i))$   
 by (pred-auto)

**lemma** *not-USUP*:  $(\neg (\bigsqcup_{i \in A} \cdot P(i))) = (\bigsqcap_{i \in A} \cdot \neg P(i))$   
 by (pred-auto)

**lemma** *not-UINF-ind*:  $(\neg (\bigsqcap i \cdot P(i))) = (\bigsqcup i \cdot \neg P(i))$   
 by (pred-auto)

**lemma** *not-USUP-ind*:  $(\neg (\bigsqcup i \cdot P(i))) = (\bigsqcap i \cdot \neg P(i))$   
 by (pred-auto)

**lemma** *UINF-empty* [simp]:  $(\bigsqcap i \in \{\} \cdot P(i)) = false$   
 by (pred-auto)

**lemma** *UINF-insert* [simp]:  $(\bigsqcap_{i \in insert\ x\ xs} \cdot P(i)) = (P(x) \sqcap (\bigsqcap_{i \in xs} \cdot P(i)))$   
 apply (pred-simp)  
 apply (subst Sup-insert[THEN sym])  
 apply (rule-tac cong[of Sup Sup])  
 apply (auto)  
 done

**lemma** *UINF-atLeast-first*:  
 $P(n) \sqcap (\bigsqcap_{i \in \{Suc\ n..\}} \cdot P(i)) = (\bigsqcap_{i \in \{n..\}} \cdot P(i))$   
**proof** –  
 have  $insert\ n\ \{Suc\ n..\} = \{n..\}$   
 by (auto)  
 thus ?thesis  
 by (metis UINF-insert)  
**qed**

**lemma** *UINF-atLeast-Suc*:  
 $(\bigsqcap_{i \in \{Suc\ m..\}} \cdot P(i)) = (\bigsqcap_{i \in \{m..\}} \cdot P(Suc\ i))$   
 by (rel-simp, metis (full-types) Suc-le-D not-less-eq-eq)

**lemma** *USUP-empty* [simp]:  $(\bigsqcup i \in \{\} \cdot P(i)) = true$   
 by (pred-auto)

**lemma** *USUP-insert* [simp]:  $(\bigsqcup_{i \in insert\ x\ xs} \cdot P(i)) = (P(x) \sqcup (\bigsqcup_{i \in xs} \cdot P(i)))$   
 apply (pred-simp)  
 apply (subst Inf-insert[THEN sym])  
 apply (rule-tac cong[of Inf Inf])  
 apply (auto)  
 done

**lemma** *USUP-atLeast-first*:

$$(P(n) \wedge (\bigsqcup i \in \{Suc\ n..\} \cdot P(i))) = (\bigsqcup i \in \{n..\} \cdot P(i))$$

**proof** –

**have**  $insert\ n\ \{Suc\ n..\} = \{n..\}$   
**by**  $(auto)$   
**thus**  $?thesis$   
**by**  $(metis\ USUP-insert\ conj-upred-def)$

**qed**

**lemma** *USUP-atLeast-Suc*:

$(\bigsqcup i \in \{Suc\ m..\} \cdot P(i)) = (\bigsqcup i \in \{m..\} \cdot P(Suc\ i))$   
**by**  $(rel-simp, metis\ (full-types)\ Suc-le-D\ not-less-eq-eq)$

**lemma** *conj-UINF-dist*:

$(P \wedge (\prod Q \in S \cdot F(Q))) = (\prod Q \in S \cdot P \wedge F(Q))$   
**by**  $(simp\ add: upred-defs\ bop.rep-eq\ lit.rep-eq, pred-auto)$

**lemma** *conj-UINF-ind-dist*:

$(P \wedge (\prod Q \cdot F(Q))) = (\prod Q \cdot P \wedge F(Q))$   
**by**  $pred-auto$

**lemma** *disj-UINF-dist*:

$S \neq \{\} \implies (P \vee (\prod Q \in S \cdot F(Q))) = (\prod Q \in S \cdot P \vee F(Q))$   
**by**  $(simp\ add: upred-defs\ bop.rep-eq\ lit.rep-eq, pred-auto)$

**lemma** *UINF-conj-UINF [simp]*:

$((\prod i \in I \cdot P(i)) \vee (\prod i \in I \cdot Q(i))) = (\prod i \in I \cdot P(i) \vee Q(i))$   
**by**  $(rel-auto)$

**lemma** *conj-USUP-dist*:

$S \neq \{\} \implies (P \wedge (\bigsqcup Q \in S \cdot F(Q))) = (\bigsqcup Q \in S \cdot P \wedge F(Q))$   
**by**  $(subst\ uexpr-eq-iff, auto\ simp\ add: conj-upred-def\ USUP.rep-eq\ inf-uexpr.rep-eq\ bop.rep-eq\ lit.rep-eq)$

**lemma** *USUP-conj-USUP [simp]*:  $((\bigsqcup P \in A \cdot F(P)) \wedge (\bigsqcup P \in A \cdot G(P))) = (\bigsqcup P \in A \cdot F(P) \wedge G(P))$

**by**  $(simp\ add: upred-defs\ bop.rep-eq\ lit.rep-eq, pred-auto)$

**lemma** *UINF-all-cong [cong]*:

**assumes**  $\bigwedge P. F(P) = G(P)$   
**shows**  $(\prod P \cdot F(P)) = (\prod P \cdot G(P))$   
**by**  $(simp\ add: UINF-as-Sup-collect\ assms)$

**lemma** *UINF-cong*:

**assumes**  $\bigwedge P. P \in A \implies F(P) = G(P)$   
**shows**  $(\prod P \in A \cdot F(P)) = (\prod P \in A \cdot G(P))$   
**by**  $(simp\ add: UINF-as-Sup-collect\ assms)$

**lemma** *USUP-all-cong*:

**assumes**  $\bigwedge P. F(P) = G(P)$   
**shows**  $(\bigsqcup P \cdot F(P)) = (\bigsqcup P \cdot G(P))$   
**by**  $(simp\ add: assms)$

**lemma** *USUP-cong*:

**assumes**  $\bigwedge P. P \in A \implies F(P) = G(P)$   
**shows**  $(\bigsqcup P \in A \cdot F(P)) = (\bigsqcup P \in A \cdot G(P))$   
**by**  $(simp\ add: USUP-as-Inf-collect\ assms)$

**lemma** *UINF-subset-mono*:  $A \subseteq B \implies (\prod_{P \in B} F(P)) \sqsubseteq (\prod_{P \in A} F(P))$   
**by** (*simp add: SUP-subset-mono UINF-as-Sup-collect*)

**lemma** *USUP-subset-mono*:  $A \subseteq B \implies (\bigsqcup_{P \in A} F(P)) \sqsubseteq (\bigsqcup_{P \in B} F(P))$   
**by** (*simp add: INF-superset-mono USUP-as-Inf-collect*)

**lemma** *UINF-impl*:  $(\prod_{P \in A} F(P) \Rightarrow G(P)) = ((\bigsqcup_{P \in A} F(P)) \Rightarrow (\prod_{P \in A} G(P)))$   
**by** (*pred-auto*)

**lemma** *USUP-is-forall*:  $(\bigsqcup x \cdot P(x)) = (\forall x \cdot P(x))$   
**by** (*pred-simp*)

**lemma** *USUP-ind-is-forall*:  $(\bigsqcup x \in A \cdot P(x)) = (\forall x \in \ll A \gg \cdot P(x))$   
**by** (*pred-auto*)

**lemma** *UINF-is-exists*:  $(\prod x \cdot P(x)) = (\exists x \cdot P(x))$   
**by** (*pred-simp*)

**lemma** *UINF-all-nats* [*simp*]:  
**fixes**  $P :: \text{nat} \Rightarrow 'a \text{ upred}$   
**shows**  $(\prod n \cdot \prod_{i \in \{0..n\}} P(i)) = (\prod n \cdot P(n))$   
**by** (*pred-auto*)

**lemma** *USUP-all-nats* [*simp*]:  
**fixes**  $P :: \text{nat} \Rightarrow 'a \text{ upred}$   
**shows**  $(\bigsqcup n \cdot \bigsqcup_{i \in \{0..n\}} P(i)) = (\bigsqcup n \cdot P(n))$   
**by** (*pred-auto*)

**lemma** *UINF-upto-expand-first*:  
 $m < n \implies (\prod_{i \in \{m..<n\}} P(i)) = ((P(m) :: 'a \text{ upred}) \vee (\prod_{i \in \{Suc\ m..<n\}} P(i)))$   
**apply** (*rel-auto*) **using** *Suc-leI le-eq-less-or-eq* **by** *auto*

**lemma** *UINF-upto-expand-last*:  
 $(\prod_{i \in \{0..<Suc(n)\}} P(i)) = ((\prod_{i \in \{0..<n\}} P(i)) \vee P(n))$   
**apply** (*rel-auto*)  
**using** *less-SucE* **by** *blast*

**lemma** *UINF-Suc-shift*:  $(\prod_{i \in \{Suc\ 0..<Suc\ n\}} P(i)) = (\prod_{i \in \{0..<n\}} P(Suc\ i))$   
**apply** (*rel-simp*)  
**apply** (*rule cong[of Sup], auto*)  
**using** *less-Suc-eq-0-disj* **by** *auto*

**lemma** *USUP-upto-expand-first*:  
 $(\bigsqcup_{i \in \{0..<Suc(n)\}} P(i)) = (P(0) \wedge (\bigsqcup_{i \in \{1..<Suc(n)\}} P(i)))$   
**apply** (*rel-auto*)  
**using** *not-less* **by** *auto*

**lemma** *USUP-Suc-shift*:  $(\bigsqcup_{i \in \{Suc\ 0..<Suc\ n\}} P(i)) = (\bigsqcup_{i \in \{0..<n\}} P(Suc\ i))$   
**apply** (*rel-simp*)  
**apply** (*rule cong[of Inf], auto*)  
**using** *less-Suc-eq-0-disj* **by** *auto*

**lemma** *UINF-list-conv*:  
 $(\prod_{i \in \{0..<length(xs)\}} f\ (xs\ !\ i)) = \text{foldr } (\vee) \ (\text{map } f\ xs)\ \text{false}$

```

apply (induct xs)
apply (rel-auto)
apply (simp add: UINF-upto-expand-first UINF-Suc-shift)
done

```

```

lemma USUP-list-conv:
  ( $\bigcup i \in \{0..<\text{length}(xs)\} \cdot f (xs ! i)$ ) = foldr ( $\wedge$ ) (map f xs) true
apply (induct xs)
apply (rel-auto)
apply (simp-all add: USUP-upto-expand-first USUP-Suc-shift)
done

```

```

lemma UINF-refines:
   $\llbracket \bigwedge i. i \in I \implies P \sqsubseteq Q i \rrbracket \implies P \sqsubseteq (\bigcap i \in I \cdot Q i)$ 
by (simp add: UINF-as-Sup-collect, metis SUP-least)

```

```

lemma UINF-refines':
  assumes  $\bigwedge i. P \sqsubseteq Q(i)$ 
shows  $P \sqsubseteq (\bigcap i \cdot Q(i))$ 
using assms
apply (rel-auto) using Sup-le-iff by fastforce

```

```

lemma UINF-pred-ueq [simp]:
   $(\bigcap x \mid \ll x \gg =_u v \cdot P(x)) = (P x) \llbracket x \rightarrow v \rrbracket$ 
by (pred-auto)

```

```

lemma UINF-pred-lit-eq [simp]:
   $(\bigcap x \mid \ll x = v \gg \cdot P(x)) = (P v)$ 
by (pred-auto)

```

### 13.3 Equality laws

```

lemma eq-upred-refl [simp]:  $(x =_u x) = \text{true}$ 
by (pred-auto)

```

```

lemma eq-upred-sym:  $(x =_u y) = (y =_u x)$ 
by (pred-auto)

```

```

lemma eq-cong-left:
  assumes vwb-lens  $x \ \$x \ \# \ Q \ \$x' \ \# \ Q \ \$x \ \# \ R \ \$x' \ \# \ R$ 
shows  $((\$x' =_u \$x \wedge Q) = (\$x' =_u \$x \wedge R)) \longleftrightarrow (Q = R)$ 
using assms
by (pred-simp, (meson mwb-lens-def vwb-lens-mwb weak-lens-def)+)

```

```

lemma conj-eq-in-var-subst:
  fixes  $x :: ('a \implies 'a)$ 
assumes vwb-lens  $x$ 
shows  $(P \wedge \$x =_u v) = (P \llbracket v / \$x \rrbracket \wedge \$x =_u v)$ 
using assms
by (pred-simp, (metis vwb-lens-wb wb-lens.get-put)+)

```

```

lemma conj-eq-out-var-subst:
  fixes  $x :: ('a \implies 'a)$ 
assumes vwb-lens  $x$ 
shows  $(P \wedge \$x' =_u v) = (P \llbracket v / \$x' \rrbracket \wedge \$x' =_u v)$ 
using assms

```

by (pred-simp, (metis vwb-lens-wb wb-lens.get-put)+)

**lemma** conj-pos-var-subst:

assumes vwb-lens x

shows  $(\$x \wedge Q) = (\$x \wedge Q[\![true/\$x]\!])$

using assms

by (pred-auto, metis (full-types) vwb-lens-wb wb-lens.get-put, metis (full-types) vwb-lens-wb wb-lens.get-put)

**lemma** conj-neg-var-subst:

assumes vwb-lens x

shows  $(\neg \$x \wedge Q) = (\neg \$x \wedge Q[\![false/\$x]\!])$

using assms

by (pred-auto, metis (full-types) vwb-lens-wb wb-lens.get-put, metis (full-types) vwb-lens-wb wb-lens.get-put)

**lemma** upred-eq-true [simp]:  $(p =_u true) = p$

by (pred-auto)

**lemma** upred-eq-false [simp]:  $(p =_u false) = (\neg p)$

by (pred-auto)

**lemma** upred-true-eq [simp]:  $(true =_u p) = p$

by (pred-auto)

**lemma** upred-false-eq [simp]:  $(false =_u p) = (\neg p)$

by (pred-auto)

**lemma** conj-var-subst:

assumes vwb-lens x

shows  $(P \wedge var\ x =_u v) = (P[\![v/x]\!] \wedge var\ x =_u v)$

using assms

by (pred-simp, (metis (full-types) vwb-lens-def wb-lens.get-put)+)

### 13.4 HOL Variable Quantifiers

**lemma** shEx-unbound [simp]:  $(\exists\ x \cdot P) = P$

by (pred-auto)

**lemma** shEx-bool [simp]:  $shEx\ P = (P\ True \vee P\ False)$

by (pred-simp, metis (full-types))

**lemma** shEx-commute:  $(\exists\ x \cdot \exists\ y \cdot P\ x\ y) = (\exists\ y \cdot \exists\ x \cdot P\ x\ y)$

by (pred-auto)

**lemma** shEx-cong:  $[\![\bigwedge x. P\ x = Q\ x]\!] \implies shEx\ P = shEx\ Q$

by (pred-auto)

**lemma** shEx-insert:  $(\exists\ x \in insert_u\ y\ A \cdot P(x)) = (P(x)[\![x \rightarrow y]\!] \vee (\exists\ x \in A \cdot P(x)))$

by (pred-auto)

**lemma** shEx-one-point:  $(\exists\ x \cdot \ll x \gg =_u v \wedge P(x)) = P(x)[\![x \rightarrow v]\!]$

by (rel-auto)

**lemma** shAll-unbound [simp]:  $(\forall\ x \cdot P) = P$

by (pred-auto)

**lemma** shAll-bool [simp]:  $shAll\ P = (P\ True \wedge P\ False)$

by (*pred-simp*, *metis* (*full-types*))

**lemma** *shAll-cong*:  $\llbracket \bigwedge x. P\ x = Q\ x \rrbracket \implies \text{shAll}\ P = \text{shAll}\ Q$   
 by (*pred-auto*)

Quantifier lifting

**named-theorems** *uquant-lift*

**lemma** *shEx-lift-conj-1* [*uquant-lift*]:  
 $((\exists x \cdot P(x)) \wedge Q) = (\exists x \cdot P(x) \wedge Q)$   
 by (*pred-auto*)

**lemma** *shEx-lift-conj-2* [*uquant-lift*]:  
 $(P \wedge (\exists x \cdot Q(x))) = (\exists x \cdot P \wedge Q(x))$   
 by (*pred-auto*)

### 13.5 Case Splitting

**lemma** *eq-split-subst*:  
 assumes *vwb-lens* *x*  
 shows  $(P = Q) \longleftrightarrow (\forall v. P\llbracket \llbracket v \rrbracket / x \rrbracket = Q\llbracket \llbracket v \rrbracket / x \rrbracket)$   
 using *assms*  
 by (*pred-auto*, *metis* *vwb-lens-wb* *wb-lens.source-stability*)

**lemma** *eq-split-substI*:  
 assumes *vwb-lens* *x*  $\bigwedge v. P\llbracket \llbracket v \rrbracket / x \rrbracket = Q\llbracket \llbracket v \rrbracket / x \rrbracket$   
 shows  $P = Q$   
 using *assms*(1) *assms*(2) *eq-split-subst* by *blast*

**lemma** *taut-split-subst*:  
 assumes *vwb-lens* *x*  
 shows  $\langle P \rangle \longleftrightarrow (\forall v. \langle P\llbracket \llbracket v \rrbracket / x \rrbracket \rangle)$   
 using *assms*  
 by (*pred-auto*, *metis* *vwb-lens-wb* *wb-lens.source-stability*)

**lemma** *eq-split*:  
 assumes  $\langle P \Rightarrow Q \rangle$   $\langle Q \Rightarrow P \rangle$   
 shows  $P = Q$   
 using *assms*  
 by (*pred-auto*)

**lemma** *bool-eq-splitI*:  
 assumes *vwb-lens* *x*  $P\llbracket \text{true}/x \rrbracket = Q\llbracket \text{true}/x \rrbracket$   $P\llbracket \text{false}/x \rrbracket = Q\llbracket \text{false}/x \rrbracket$   
 shows  $P = Q$   
 by (*metis* (*full-types*) *assms* *eq-split-subst* *false-alt-def* *true-alt-def*)

**lemma** *subst-bool-split*:  
 assumes *vwb-lens* *x*  
 shows  $\langle P \rangle = \langle (P\llbracket \text{false}/x \rrbracket \wedge P\llbracket \text{true}/x \rrbracket) \rangle$   
**proof** –  
 from *assms* have  $\langle P \rangle = (\forall v. \langle P\llbracket \llbracket v \rrbracket / x \rrbracket \rangle)$   
 by (*subst* *taut-split-subst*[*of* *x*], *auto*)  
 also have  $\dots = \langle P\llbracket \llbracket \text{True} \rrbracket / x \rrbracket \wedge P\llbracket \llbracket \text{False} \rrbracket / x \rrbracket \rangle$   
 by (*metis* (*mono-tags*, *lifting*))  
 also have  $\dots = \langle (P\llbracket \text{false}/x \rrbracket \wedge P\llbracket \text{true}/x \rrbracket) \rangle$   
 by (*pred-auto*)

finally show *?thesis* .  
qed

**lemma** *subst-eq-replace*:  
 fixes  $x :: ('a \implies 'a)$   
 shows  $(p[u/x] \wedge u =_u v) = (p[v/x] \wedge u =_u v)$   
 by (*pred-auto*)

## 13.6 UTP Quantifiers

**lemma** *one-point*:  
 assumes *mbw-lens*  $x \not\# v$   
 shows  $(\exists x \cdot P \wedge \text{var } x =_u v) = P[v/x]$   
 using *assms*  
 by (*pred-auto*)

**lemma** *exists-twice*: *mbw-lens*  $x \implies (\exists x \cdot \exists x \cdot P) = (\exists x \cdot P)$   
 by (*pred-auto*)

**lemma** *all-twice*: *mbw-lens*  $x \implies (\forall x \cdot \forall x \cdot P) = (\forall x \cdot P)$   
 by (*pred-auto*)

**lemma** *exists-sub*:  $\llbracket \text{mbw-lens } y; x \subseteq_L y \rrbracket \implies (\exists x \cdot \exists y \cdot P) = (\exists y \cdot P)$   
 by (*pred-auto*)

**lemma** *all-sub*:  $\llbracket \text{mbw-lens } y; x \subseteq_L y \rrbracket \implies (\forall x \cdot \forall y \cdot P) = (\forall y \cdot P)$   
 by (*pred-auto*)

**lemma** *ex-commute*:  
 assumes  $x \bowtie y$   
 shows  $(\exists x \cdot \exists y \cdot P) = (\exists y \cdot \exists x \cdot P)$   
 using *assms*  
 apply (*pred-auto*)  
 using *lens-indep-comm* **apply** *fastforce* +  
 done

**lemma** *all-commute*:  
 assumes  $x \bowtie y$   
 shows  $(\forall x \cdot \forall y \cdot P) = (\forall y \cdot \forall x \cdot P)$   
 using *assms*  
 apply (*pred-auto*)  
 using *lens-indep-comm* **apply** *fastforce* +  
 done

**lemma** *ex-equiv*:  
 assumes  $x \approx_L y$   
 shows  $(\exists x \cdot P) = (\exists y \cdot P)$   
 using *assms*  
 by (*pred-simp*, *metis* (*no-types*, *lifting*) *lens.select-convs*(2))

**lemma** *all-equiv*:  
 assumes  $x \approx_L y$   
 shows  $(\forall x \cdot P) = (\forall y \cdot P)$   
 using *assms*  
 by (*pred-simp*, *metis* (*no-types*, *lifting*) *lens.select-convs*(2))

**lemma** *ex-zero*:

$(\exists \emptyset \cdot P) = P$   
**by** (*pred-auto*)

**lemma** *all-zero*:

$(\forall \emptyset \cdot P) = P$   
**by** (*pred-auto*)

**lemma** *ex-plus*:

$(\exists y; x \cdot P) = (\exists x \cdot \exists y \cdot P)$   
**by** (*pred-auto*)

**lemma** *all-plus*:

$(\forall y; x \cdot P) = (\forall x \cdot \forall y \cdot P)$   
**by** (*pred-auto*)

**lemma** *closure-all*:

$[P]_u = (\forall \Sigma \cdot P)$   
**by** (*pred-auto*)

**lemma** *unrest-as-exists*:

$vwb\text{-}lens\ x \implies (x \# P) \longleftrightarrow ((\exists x \cdot P) = P)$   
**by** (*pred-simp*, *metis vwb-lens.put-eq*)

**lemma** *ex-mono*:  $P \sqsubseteq Q \implies (\exists x \cdot P) \sqsubseteq (\exists x \cdot Q)$

**by** (*pred-auto*)

**lemma** *ex-weakens*:  $wb\text{-}lens\ x \implies (\exists x \cdot P) \sqsubseteq P$

**by** (*pred-simp*, *metis wb-lens.get-put*)

**lemma** *all-mono*:  $P \sqsubseteq Q \implies (\forall x \cdot P) \sqsubseteq (\forall x \cdot Q)$

**by** (*pred-auto*)

**lemma** *all-strengthens*:  $wb\text{-}lens\ x \implies P \sqsubseteq (\forall x \cdot P)$

**by** (*pred-simp*, *metis wb-lens.get-put*)

**lemma** *ex-unrest*:  $x \# P \implies (\exists x \cdot P) = P$

**by** (*pred-auto*)

**lemma** *all-unrest*:  $x \# P \implies (\forall x \cdot P) = P$

**by** (*pred-auto*)

**lemma** *not-ex-not*:  $\neg (\exists x \cdot \neg P) = (\forall x \cdot P)$

**by** (*pred-auto*)

**lemma** *not-all-not*:  $\neg (\forall x \cdot \neg P) = (\exists x \cdot P)$

**by** (*pred-auto*)

**lemma** *ex-conj-contr-left*:  $x \# P \implies (\exists x \cdot P \wedge Q) = (P \wedge (\exists x \cdot Q))$

**by** (*pred-auto*)

**lemma** *ex-conj-contr-right*:  $x \# Q \implies (\exists x \cdot P \wedge Q) = ((\exists x \cdot P) \wedge Q)$

**by** (*pred-auto*)



## 13.7 Variable Restriction

**lemma** *var-res-all*:

$P \upharpoonright_v \Sigma = P$   
**by** (*rel-auto*)

**lemma** *var-res-twice*:

$mwb\text{-}lens\ x \implies P \upharpoonright_v x \upharpoonright_v x = P \upharpoonright_v x$   
**by** (*pred-auto*)

## 13.8 Conditional laws

**lemma** *cond-def*:

$(P \triangleleft b \triangleright Q) = ((b \wedge P) \vee ((\neg b) \wedge Q))$   
**by** (*pred-auto*)

**lemma** *cond-idem* [*simp*]:  $(P \triangleleft b \triangleright P) = P$  **by** (*pred-auto*)

**lemma** *cond-true-false* [*simp*]:  $true \triangleleft b \triangleright false = b$  **by** (*pred-auto*)

**lemma** *cond-symm*:  $(P \triangleleft b \triangleright Q) = (Q \triangleleft \neg b \triangleright P)$  **by** (*pred-auto*)

**lemma** *cond-assoc*:  $((P \triangleleft b \triangleright Q) \triangleleft c \triangleright R) = (P \triangleleft b \wedge c \triangleright (Q \triangleleft c \triangleright R))$  **by** (*pred-auto*)

**lemma** *cond-distr*:  $(P \triangleleft b \triangleright (Q \triangleleft c \triangleright R)) = ((P \triangleleft b \triangleright Q) \triangleleft c \triangleright (P \triangleleft b \triangleright R))$  **by** (*pred-auto*)

**lemma** *cond-unit-T* [*simp*]:  $(P \triangleleft true \triangleright Q) = P$  **by** (*pred-auto*)

**lemma** *cond-unit-F* [*simp*]:  $(P \triangleleft false \triangleright Q) = Q$  **by** (*pred-auto*)

**lemma** *cond-conj-not*:  $((P \triangleleft b \triangleright Q) \wedge (\neg b)) = (Q \wedge (\neg b))$   
**by** (*rel-auto*)

**lemma** *cond-and-T-integrate*:

$((P \wedge b) \vee (Q \triangleleft b \triangleright R)) = ((P \vee Q) \triangleleft b \triangleright R)$   
**by** (*pred-auto*)

**lemma** *cond-L6*:  $(P \triangleleft b \triangleright (Q \triangleleft b \triangleright R)) = (P \triangleleft b \triangleright R)$  **by** (*pred-auto*)

**lemma** *cond-L7*:  $(P \triangleleft b \triangleright (P \triangleleft c \triangleright Q)) = (P \triangleleft b \vee c \triangleright Q)$  **by** (*pred-auto*)

**lemma** *cond-and-distr*:  $((P \wedge Q) \triangleleft b \triangleright (R \wedge S)) = ((P \triangleleft b \triangleright R) \wedge (Q \triangleleft b \triangleright S))$  **by** (*pred-auto*)

**lemma** *cond-or-distr*:  $((P \vee Q) \triangleleft b \triangleright (R \vee S)) = ((P \triangleleft b \triangleright R) \vee (Q \triangleleft b \triangleright S))$  **by** (*pred-auto*)

**lemma** *cond-imp-distr*:

$((P \implies Q) \triangleleft b \triangleright (R \implies S)) = ((P \triangleleft b \triangleright R) \implies (Q \triangleleft b \triangleright S))$  **by** (*pred-auto*)

**lemma** *cond-eq-distr*:

$((P \Leftrightarrow Q) \triangleleft b \triangleright (R \Leftrightarrow S)) = ((P \triangleleft b \triangleright R) \Leftrightarrow (Q \triangleleft b \triangleright S))$  **by** (*pred-auto*)

**lemma** *cond-conj-distr*:  $(P \wedge (Q \triangleleft b \triangleright S)) = ((P \wedge Q) \triangleleft b \triangleright (P \wedge S))$  **by** (*pred-auto*)

**lemma** *cond-disj-distr*:  $(P \vee (Q \triangleleft b \triangleright S)) = ((P \vee Q) \triangleleft b \triangleright (P \vee S))$  **by** (*pred-auto*)

**lemma** *cond-neg*:  $\neg (P \triangleleft b \triangleright Q) = ((\neg P) \triangleleft b \triangleright (\neg Q))$  **by** (*pred-auto*)

**lemma** *cond-conj*:  $P \triangleleft b \wedge c \triangleright Q = (P \triangleleft c \triangleright Q) \triangleleft b \triangleright Q$   
**by** (*pred-auto*)

**lemma** *spec-cond-dist*:  $(P \Rightarrow (Q \triangleleft b \triangleright R)) = ((P \Rightarrow Q) \triangleleft b \triangleright (P \Rightarrow R))$   
**by** (*pred-auto*)

**lemma** *cond-USUP-dist*:  $(\bigsqcup P \in S \cdot F(P)) \triangleleft b \triangleright (\bigsqcup P \in S \cdot G(P)) = (\bigsqcup P \in S \cdot F(P) \triangleleft b \triangleright G(P))$   
**by** (*pred-auto*)

**lemma** *cond-UINF-dist*:  $(\bigsqcap P \in S \cdot F(P)) \triangleleft b \triangleright (\bigsqcap P \in S \cdot G(P)) = (\bigsqcap P \in S \cdot F(P) \triangleleft b \triangleright G(P))$   
**by** (*pred-auto*)

**lemma** *cond-var-subst-left*:  
**assumes** *vwb-lens x*  
**shows**  $(P \llbracket \text{true}/x \rrbracket \triangleleft \text{var } x \triangleright Q) = (P \triangleleft \text{var } x \triangleright Q)$   
**using** *assms* **by** (*pred-auto*, *metis (full-types) vwb-lens-wb wb-lens.get-put*)

**lemma** *cond-var-subst-right*:  
**assumes** *vwb-lens x*  
**shows**  $(P \triangleleft \text{var } x \triangleright Q \llbracket \text{false}/x \rrbracket) = (P \triangleleft \text{var } x \triangleright Q)$   
**using** *assms* **by** (*pred-auto*, *metis (full-types) vwb-lens.put-eq*)

**lemma** *cond-var-split*:  
*vwb-lens x*  $\implies (P \llbracket \text{true}/x \rrbracket \triangleleft \text{var } x \triangleright P \llbracket \text{false}/x \rrbracket) = P$   
**by** (*rel-simp*, (*metis (full-types) vwb-lens.put-eq*)+)

**lemma** *cond-assign-subst*:  
*vwb-lens x*  $\implies (P \triangleleft \text{utp-expr.var } x =_u v \triangleright Q) = (P \llbracket v/x \rrbracket \triangleleft \text{utp-expr.var } x =_u v \triangleright Q)$   
**apply** (*rel-simp*) **using** *vwb-lens.put-eq* **by** *force*

**lemma** *conj-conds*:  
 $(P1 \triangleleft b \triangleright Q1 \wedge P2 \triangleleft b \triangleright Q2) = (P1 \wedge P2) \triangleleft b \triangleright (Q1 \wedge Q2)$   
**by** *pred-auto*

**lemma** *disj-conds*:  
 $(P1 \triangleleft b \triangleright Q1 \vee P2 \triangleleft b \triangleright Q2) = (P1 \vee P2) \triangleleft b \triangleright (Q1 \vee Q2)$   
**by** *pred-auto*

**lemma** *cond-mono*:  
 $\llbracket P1 \sqsubseteq P2; Q1 \sqsubseteq Q2 \rrbracket \implies (P1 \triangleleft b \triangleright Q1) \sqsubseteq (P2 \triangleleft b \triangleright Q2)$   
**by** (*rel-auto*)

**lemma** *cond-monotonic*:  
 $\llbracket \text{mono } P; \text{mono } Q \rrbracket \implies \text{mono } (\lambda X. P X \triangleleft b \triangleright Q X)$   
**by** (*simp add: mono-def, rel-blast*)

## 13.9 Additional Expression Laws

**lemma** *le-pred-refl [simp]*:  
**fixes**  $x :: ('a::\text{preorder}, 'a) \text{ uexpr}$   
**shows**  $(x \leq_u x) = \text{true}$   
**by** (*pred-auto*)

**lemma** *uzero-le-laws [simp]*:  
 $(0 :: ('a::\{\text{linordered-semidom}\}, 'a) \text{ uexpr}) \leq_u \text{numeral } x = \text{true}$

$(1 :: ('a::\{\text{linordered-semidom}\}, 'a) \text{ uexpr}) \leq_u \text{numeral } x = \text{true}$   
 $(0 :: ('a::\{\text{linordered-semidom}\}, 'a) \text{ uexpr}) \leq_u 1 = \text{true}$   
**by** (*pred-simp*)+

**lemma** *unumeral-le-1* [*simp*]:  
**assumes** ( $\text{numeral } i :: 'a::\{\text{numeral,ord}\} \leq \text{numeral } j$ )  
**shows** ( $\text{numeral } i :: ('a, 'a) \text{ uexpr}) \leq_u \text{numeral } j = \text{true}$   
**using** *assms* **by** (*pred-auto*)

**lemma** *unumeral-le-2* [*simp*]:  
**assumes** ( $\text{numeral } i :: 'a::\{\text{numeral,linorder}\} > \text{numeral } j$ )  
**shows** ( $\text{numeral } i :: ('a, 'a) \text{ uexpr}) \leq_u \text{numeral } j = \text{false}$   
**using** *assms* **by** (*pred-auto*)

**lemma** *uset-laws* [*simp*]:  
 $x \in_u \{\}_u = \text{false}$   
 $x \in_u \{m..n\}_u = (m \leq_u x \wedge x \leq_u n)$   
**by** (*pred-auto*)+

**lemma** *ulit-eq* [*simp*]:  $x = y \implies (\llbracket x \rrbracket =_u \llbracket y \rrbracket) = \text{true}$   
**by** (*rel-auto*)

**lemma** *ulit-neq* [*simp*]:  $x \neq y \implies (\llbracket x \rrbracket =_u \llbracket y \rrbracket) = \text{false}$   
**by** (*rel-auto*)

**lemma** *uset-mems* [*simp*]:  
 $x \in_u \{y\}_u = (x =_u y)$   
 $x \in_u A \cup_u B = (x \in_u A \vee x \in_u B)$   
 $x \in_u A \cap_u B = (x \in_u A \wedge x \in_u B)$   
**by** (*rel-auto*)+

## 13.10 Refinement By Observation

Function to obtain the set of observations of a predicate

**definition** *obs-upred* ::  $'a \text{ upred} \Rightarrow 'a \text{ set } ([\_])_o$   
**where** [*upred-defs*]:  $[\![P]\!]_o = \{b. [\![P]\!]_e b\}$

**lemma** *obs-upred-refine-iff*:  
 $P \sqsubseteq Q \iff [\![Q]\!]_o \subseteq [\![P]\!]_o$   
**by** (*pred-auto*)

A refinement can be demonstrated by considering only the observations of the predicates which are relevant, i.e. not unrestricted, for them. In other words, if the alphabet can be split into two disjoint segments,  $x$  and  $y$ , and neither predicate refers to  $y$  then only  $x$  need be considered when checking for observations.

**lemma** *refine-by-obs*:  
**assumes**  $x \bowtie y \text{ bij-lens } (x +_L y) \ y \# P \ y \# Q \ \{v. 'P[\![\llbracket v \rrbracket/x]\!]\} \subseteq \{v. 'Q[\![\llbracket v \rrbracket/x]\!]\}$   
**shows**  $Q \sqsubseteq P$   
**using** *assms*(3–5)  
**apply** (*simp add: obs-upred-refine-iff subset-eq*)  
**apply** (*pred-simp*)  
**apply** (*rename-tac b*)  
**apply** (*drule-tac x=get\_x b in spec*)  
**apply** (*auto simp add: assms*)

**apply** (*metis* *assms*(1) *assms*(2) *bij-lens.axioms*(2) *bij-lens-axioms-def* *lens-override-def* *lens-override-plus*)+  
**done**

### 13.11 Cylindric Algebra

**lemma** *C1*:  $(\exists x \cdot \text{false}) = \text{false}$   
**by** (*pred-auto*)

**lemma** *C2*:  $\text{wb-lens } x \implies 'P \Rightarrow (\exists x \cdot P)'$   
**by** (*pred-simp*, *metis* *wb-lens.get-put*)

**lemma** *C3*:  $\text{mwb-lens } x \implies (\exists x \cdot (P \wedge (\exists x \cdot Q))) = ((\exists x \cdot P) \wedge (\exists x \cdot Q))$   
**by** (*pred-auto*)

**lemma** *C4a*:  $x \approx_L y \implies (\exists x \cdot \exists y \cdot P) = (\exists y \cdot \exists x \cdot P)$   
**by** (*pred-simp*, *metis* (*no-types*, *lifting*) *lens.select-convs*(2))+

**lemma** *C4b*:  $x \bowtie y \implies (\exists x \cdot \exists y \cdot P) = (\exists y \cdot \exists x \cdot P)$   
**using** *ex-commute* **by** *blast*

**lemma** *C5*:  
**fixes**  $x :: ('a \implies 'a)$   
**shows**  $(\&x =_u \&x) = \text{true}$   
**by** (*pred-auto*)

**lemma** *C6*:  
**assumes**  $\text{wb-lens } x \bowtie y \bowtie z$   
**shows**  $(\&y =_u \&z) = (\exists x \cdot \&y =_u \&x \wedge \&x =_u \&z)$   
**using** *assms*  
**by** (*pred-simp*, (*metis* *lens-indep-def*)+)

**lemma** *C7*:  
**assumes**  $\text{weak-lens } x \bowtie y$   
**shows**  $((\exists x \cdot \&x =_u \&y \wedge P) \wedge (\exists x \cdot \&x =_u \&y \wedge \neg P)) = \text{false}$   
**using** *assms*  
**by** (*pred-simp*, *simp* *add: lens-indep-sym*)

**end**

## 14 Healthiness Conditions

**theory** *utp-healthy*  
**imports** *utp-pred-laws*  
**begin**

### 14.1 Main Definitions

We collect closure laws for healthiness conditions in the following theorem attribute.

**named-theorems** *closure*

**type-synonym**  $'\alpha \text{ health} = '\alpha \text{ upred} \Rightarrow '\alpha \text{ upred}$

A predicate  $P$  is healthy, under healthiness function  $H$ , if  $P$  is a fixed-point of  $H$ .

**definition** *Healthy* ::  $'\alpha \text{ upred} \Rightarrow '\alpha \text{ health} \Rightarrow \text{bool}$  (**infix** *is* 30)

where  $P \text{ is } H \equiv (H \ P = P)$

**lemma** *Healthy-def'*:  $P \text{ is } H \longleftrightarrow (H \ P = P)$   
**unfolding** *Healthy-def* **by** *auto*

**lemma** *Healthy-if*:  $P \text{ is } H \implies (H \ P = P)$   
**unfolding** *Healthy-def* **by** *auto*

**lemma** *Healthy-intro*:  $H(P) = P \implies P \text{ is } H$   
**by** (*simp add: Healthy-def*)

**declare** *Healthy-def'* [*upred-defs*]

**abbreviation** *Healthy-carrier* ::  $'\alpha \text{ health} \Rightarrow '\alpha \text{ upred set } (\llbracket - \rrbracket_H)$   
**where**  $\llbracket H \rrbracket_H \equiv \{P. P \text{ is } H\}$

**lemma** *Healthy-carrier-image*:  
 $A \subseteq \llbracket \mathcal{H} \rrbracket_H \implies \mathcal{H} \ ' A = A$   
**by** (*auto simp add: image-def, (metis Healthy-if mem-Collect-eq subsetCE)+*)

**lemma** *Healthy-carrier-Collect*:  $A \subseteq \llbracket H \rrbracket_H \implies A = \{H(P) \mid P. P \in A\}$   
**by** (*simp add: Healthy-carrier-image Setcompr-eq-image*)

**lemma** *Healthy-func*:  
 $\llbracket F \in \llbracket \mathcal{H}_1 \rrbracket_H \rightarrow \llbracket \mathcal{H}_2 \rrbracket_H; P \text{ is } \mathcal{H}_1 \rrbracket \implies \mathcal{H}_2(F(P)) = F(P)$   
**using** *Healthy-if* **by** *blast*

**lemma** *Healthy-comp*:  
 $\llbracket P \text{ is } \mathcal{H}_1; P \text{ is } \mathcal{H}_2 \rrbracket \implies P \text{ is } \mathcal{H}_1 \circ \mathcal{H}_2$   
**by** (*simp add: Healthy-def*)

**lemma** *Healthy-apply-closed*:  
**assumes**  $F \in \llbracket H \rrbracket_H \rightarrow \llbracket H \rrbracket_H \ P \text{ is } H$   
**shows**  $F(P) \text{ is } H$   
**using** *assms(1) assms(2)* **by** *auto*

**lemma** *Healthy-set-image-member*:  
 $\llbracket P \in F \ ' A; \bigwedge x. F \ x \text{ is } H \rrbracket \implies P \text{ is } H$   
**by** *blast*

**lemma** *Healthy-case-prod [closure]*:  
 $\llbracket \bigwedge x \ y. P \ x \ y \text{ is } H \rrbracket \implies \text{case-prod } P \ v \text{ is } H$   
**by** (*simp add: prod.case-eq-if*)

**lemma** *Healthy-SUPREMUM*:  
 $A \subseteq \llbracket H \rrbracket_H \implies \text{SUPREMUM } A \ H = \bigcap A$   
**by** (*drule Healthy-carrier-image, presburger*)

**lemma** *Healthy-INFIMUM*:  
 $A \subseteq \llbracket H \rrbracket_H \implies \text{INFIMUM } A \ H = \bigcup A$   
**by** (*drule Healthy-carrier-image, presburger*)

**lemma** *Healthy-nu [closure]*:  
**assumes**  $\text{mono } F \ F \in \llbracket id \rrbracket_H \rightarrow \llbracket H \rrbracket_H$   
**shows**  $\nu \ F \text{ is } H$

by (metis (mono-tags) Healthy-def Healthy-func assms eq-id-iff lfp-unfold)

**lemma** *Healthy-mu* [closure]:

assumes *mono*  $F \ F \in \llbracket id \rrbracket_H \rightarrow \llbracket H \rrbracket_H$

shows  $\mu \ F$  is  $H$

by (metis (mono-tags) Healthy-def Healthy-func assms eq-id-iff gfp-unfold)

**lemma** *Healthy-subset-member*:  $\llbracket A \subseteq \llbracket H \rrbracket_H; P \in A \rrbracket \implies H(P) = P$

by (meson Ball-Collect Healthy-if)

**lemma** *is-Healthy-subset-member*:  $\llbracket A \subseteq \llbracket H \rrbracket_H; P \in A \rrbracket \implies P$  is  $H$

by blast

## 14.2 Properties of Healthiness Conditions

**definition** *Idempotent* ::  $'\alpha$  health  $\Rightarrow$  bool **where**

$Idempotent(H) \longleftrightarrow (\forall P. H(H(P)) = H(P))$

**abbreviation** *Monotonic* ::  $'\alpha$  health  $\Rightarrow$  bool **where**

$Monotonic(H) \equiv mono \ H$

**definition** *IMH* ::  $'\alpha$  health  $\Rightarrow$  bool **where**

$IMH(H) \longleftrightarrow Idempotent(H) \wedge Monotonic(H)$

**definition** *Antitone* ::  $'\alpha$  health  $\Rightarrow$  bool **where**

$Antitone(H) \longleftrightarrow (\forall P \ Q. Q \sqsubseteq P \longrightarrow (H(P) \sqsubseteq H(Q)))$

**definition** *Conjunctive* ::  $'\alpha$  health  $\Rightarrow$  bool **where**

$Conjunctive(H) \longleftrightarrow (\exists Q. \forall P. H(P) = (P \wedge Q))$

**definition** *FunctionalConjunctive* ::  $'\alpha$  health  $\Rightarrow$  bool **where**

$FunctionalConjunctive(H) \longleftrightarrow (\exists F. \forall P. H(P) = (P \wedge F(P)) \wedge Monotonic(F))$

**definition** *WeakConjunctive* ::  $'\alpha$  health  $\Rightarrow$  bool **where**

$WeakConjunctive(H) \longleftrightarrow (\forall P. \exists Q. H(P) = (P \wedge Q))$

**definition** *Disjunctuous* ::  $'\alpha$  health  $\Rightarrow$  bool **where**

[upred-defs]:  $Disjunctuous \ H = (\forall P \ Q. H(P \sqcap Q) = (H(P) \sqcap H(Q)))$

**definition** *Continuous* ::  $'\alpha$  health  $\Rightarrow$  bool **where**

[upred-defs]:  $Continuous \ H = (\forall A. A \neq \{\} \longrightarrow H(\bigsqcap A) = \bigsqcap (H \restriction A))$

**lemma** *Healthy-Idempotent* [closure]:

$Idempotent \ H \implies H(P)$  is  $H$

by (simp add: Healthy-def Idempotent-def)

**lemma** *Healthy-range*:  $Idempotent \ H \implies range \ H = \llbracket H \rrbracket_H$

by (auto simp add: image-def Healthy-if Healthy-Idempotent, metis Healthy-if)

**lemma** *Idempotent-id* [simp]:  $Idempotent \ id$

by (simp add: Idempotent-def)

**lemma** *Idempotent-comp* [intro]:

$\llbracket Idempotent \ f; Idempotent \ g; f \circ g = g \circ f \rrbracket \implies Idempotent \ (f \circ g)$

by (auto simp add: Idempotent-def comp-def, metis)

**lemma** *Idempotent-image*:  $\text{Idempotent } f \implies f \circ f \circ A = f \circ A$   
**by** (*metis* (*mono-tags*, *lifting*) *Idempotent-def image-cong image-image*)

**lemma** *Monotonic-id* [*simp*]: *Monotonic id*  
**by** (*simp* *add*: *monoI*)

**lemma** *Monotonic-id'* [*closure*]:  
*mono* ( $\lambda X. X$ )  
**by** (*simp* *add*: *monoI*)

**lemma** *Monotonic-const* [*closure*]:  
*Monotonic* ( $\lambda x. c$ )  
**by** (*simp* *add*: *mono-def*)

**lemma** *Monotonic-comp* [*intro*]:  
 $\llbracket \text{Monotonic } f; \text{Monotonic } g \rrbracket \implies \text{Monotonic } (f \circ g)$   
**by** (*simp* *add*: *mono-def*)

**lemma** *Monotonic-inf* [*closure*]:  
**assumes** *Monotonic P Monotonic Q*  
**shows** *Monotonic* ( $\lambda X. P(X) \sqcap Q(X)$ )  
**using** *assms* **by** (*simp* *add*: *mono-def*, *rel-auto*)

**lemma** *Monotonic-cond* [*closure*]:  
**assumes** *Monotonic P Monotonic Q*  
**shows** *Monotonic* ( $\lambda X. P(X) \triangleleft b \triangleright Q(X)$ )  
**by** (*simp* *add*: *assms cond-monotonic*)

**lemma** *Conjunctive-Idempotent*:  
 $\text{Conjunctive}(H) \implies \text{Idempotent}(H)$   
**by** (*auto* *simp* *add*: *Conjunctive-def Idempotent-def*)

**lemma** *Conjunctive-Monotonic*:  
 $\text{Conjunctive}(H) \implies \text{Monotonic}(H)$   
**unfolding** *Conjunctive-def mono-def*  
**using** *dual-order.trans* **by** *fastforce*

**lemma** *Conjunctive-conj*:  
**assumes** *Conjunctive(HC)*  
**shows**  $HC(P \wedge Q) = (HC(P) \wedge Q)$   
**using** *assms* **unfolding** *Conjunctive-def*  
**by** (*metis* *utp-pred-laws.inf.assoc utp-pred-laws.inf.commute*)

**lemma** *Conjunctive-distr-conj*:  
**assumes** *Conjunctive(HC)*  
**shows**  $HC(P \wedge Q) = (HC(P) \wedge HC(Q))$   
**using** *assms* **unfolding** *Conjunctive-def*  
**by** (*metis* *Conjunctive-conj assms utp-pred-laws.inf.assoc utp-pred-laws.inf-right-idem*)

**lemma** *Conjunctive-distr-disj*:  
**assumes** *Conjunctive(HC)*  
**shows**  $HC(P \vee Q) = (HC(P) \vee HC(Q))$   
**using** *assms* **unfolding** *Conjunctive-def*  
**using** *utp-pred-laws.inf-sup-distrib2* **by** *fastforce*

**lemma** *Conjunctive-distr-cond*:  
**assumes** *Conjunctive*( $HC$ )  
**shows**  $HC(P \triangleleft b \triangleright Q) = (HC(P) \triangleleft b \triangleright HC(Q))$   
**using** *assms* **unfolding** *Conjunctive-def*  
**by** (*metis cond-conj-distr utp-pred-laws.inf-commute*)

**lemma** *FunctionalConjunctive-Monotonic*:  
 $FunctionalConjunctive(H) \implies Monotonic(H)$   
**unfolding** *FunctionalConjunctive-def* **by** (*metis mono-def utp-pred-laws.inf-mono*)

**lemma** *WeakConjunctive-Refinement*:  
**assumes** *WeakConjunctive*( $HC$ )  
**shows**  $P \sqsubseteq HC(P)$   
**using** *assms* **unfolding** *WeakConjunctive-def* **by** (*metis utp-pred-laws.inf.cobounded1*)

**lemma** *WeakConjunctive-Healthy-Refinement*:  
**assumes** *WeakConjunctive*( $HC$ ) **and**  $P$  is  $HC$   
**shows**  $HC(P) \sqsubseteq P$   
**using** *assms* **unfolding** *WeakConjunctive-def Healthy-def* **by** *simp*

**lemma** *WeakConjunctive-implies-WeakConjunctive*:  
 $Conjunctive(H) \implies WeakConjunctive(H)$   
**unfolding** *WeakConjunctive-def Conjunctive-def* **by** *pred-auto*

**declare** *Conjunctive-def* [*upred-defs*]  
**declare** *mono-def* [*upred-defs*]

**lemma** *Disjunctuous-Monotonic*:  $Disjunctuous H \implies Monotonic H$   
**by** (*metis Disjunctuous-def mono-def semilattice-sup-class.le-iff-sup*)

**lemma** *ContinuousD* [*dest*]:  $\llbracket Continuous H; A \neq \{\} \rrbracket \implies H (\bigsqcap A) = (\bigsqcap_{P \in A} H(P))$   
**by** (*simp add: Continuous-def*)

**lemma** *Continuous-Disjunctuous*:  $Continuous H \implies Disjunctuous H$   
**apply** (*auto simp add: Continuous-def Disjunctuous-def*)  
**apply** (*rename-tac P Q*)  
**apply** (*drule-tac x={P,Q} in spec*)  
**apply** (*simp*)  
**done**

**lemma** *Continuous-Monotonic* [*closure*]:  $Continuous H \implies Monotonic H$   
**by** (*simp add: Continuous-Disjunctuous Disjunctuous-Monotonic*)

**lemma** *Continuous-comp* [*intro*]:  
 $\llbracket Continuous f; Continuous g \rrbracket \implies Continuous (f \circ g)$   
**by** (*simp add: Continuous-def*)

**lemma** *Continuous-const* [*closure*]:  $Continuous (\lambda X. P)$   
**by** *pred-auto*

**lemma** *Continuous-cond* [*closure*]:  
**assumes**  $Continuous F$   $Continuous G$   
**shows**  $Continuous (\lambda X. F(X) \triangleleft b \triangleright G(X))$   
**using** *assms* **by** (*pred-auto*)

Closure laws derived from continuity



**lemma** *Sup-Continuous-closed* [closure]:

$\llbracket \text{Continuous } H; \bigwedge i. i \in A \implies P(i) \text{ is } H; A \neq \{\} \rrbracket \implies (\bigcap_{i \in A}. P(i)) \text{ is } H$   
**by** (*drule ContinuousD*[of  $H \ P \ 'A$ ], *simp add: UINF-mem-UNIV*[*THEN sym*] *UINF-as-Sup*[*THEN sym*])  
 (*metis* (*no-types*, *lifting*) *Healthy-def' SUP-cong image-image*)

**lemma** *UINF-mem-Continuous-closed* [closure]:

$\llbracket \text{Continuous } H; \bigwedge i. i \in A \implies P(i) \text{ is } H; A \neq \{\} \rrbracket \implies (\bigcap_{i \in A}. P(i)) \text{ is } H$   
**by** (*simp add: Sup-Continuous-closed UINF-as-Sup-collect*)

**lemma** *UINF-mem-Continuous-closed-pair* [closure]:

**assumes** *Continuous*  $H \bigwedge i j. (i, j) \in A \implies P \ i \ j \text{ is } H \ A \neq \{\}$   
**shows**  $(\bigcap_{(i,j) \in A}. P \ i \ j) \text{ is } H$

**proof** –

**have**  $(\bigcap_{(i,j) \in A}. P \ i \ j) = (\bigcap_{x \in A}. P \ (\text{fst } x) \ (\text{snd } x))$   
**by** (*rel-auto*)

**also have** ... *is*  $H$

**by** (*metis* (*mono-tags*) *UINF-mem-Continuous-closed assms(1) assms(2) assms(3) prod.collapse*)  
**finally show** *?thesis* .

**qed**

**lemma** *UINF-mem-Continuous-closed-triple* [closure]:

**assumes** *Continuous*  $H \bigwedge i j k. (i, j, k) \in A \implies P \ i \ j \ k \text{ is } H \ A \neq \{\}$   
**shows**  $(\bigcap_{(i,j,k) \in A}. P \ i \ j \ k) \text{ is } H$

**proof** –

**have**  $(\bigcap_{(i,j,k) \in A}. P \ i \ j \ k) = (\bigcap_{x \in A}. P \ (\text{fst } x) \ (\text{fst } (\text{snd } x)) \ (\text{snd } (\text{snd } x)))$   
**by** (*rel-auto*)

**also have** ... *is*  $H$

**by** (*metis* (*mono-tags*) *UINF-mem-Continuous-closed assms(1) assms(2) assms(3) prod.collapse*)  
**finally show** *?thesis* .

**qed**

**lemma** *UINF-mem-Continuous-closed-quad* [closure]:

**assumes** *Continuous*  $H \bigwedge i j k l. (i, j, k, l) \in A \implies P \ i \ j \ k \ l \text{ is } H \ A \neq \{\}$   
**shows**  $(\bigcap_{(i,j,k,l) \in A}. P \ i \ j \ k \ l) \text{ is } H$

**proof** –

**have**  $(\bigcap_{(i,j,k,l) \in A}. P \ i \ j \ k \ l) = (\bigcap_{x \in A}. P \ (\text{fst } x) \ (\text{fst } (\text{snd } x)) \ (\text{fst } (\text{snd } (\text{snd } x))) \ (\text{snd } (\text{snd } (\text{snd } x))))$   
**by** (*rel-auto*)

**also have** ... *is*  $H$

**by** (*metis* (*mono-tags*) *UINF-mem-Continuous-closed assms(1) assms(2) assms(3) prod.collapse*)  
**finally show** *?thesis* .

**qed**

**lemma** *UINF-mem-Continuous-closed-quint* [closure]:

**assumes** *Continuous*  $H \bigwedge i j k l m. (i, j, k, l, m) \in A \implies P \ i \ j \ k \ l \ m \text{ is } H \ A \neq \{\}$   
**shows**  $(\bigcap_{(i,j,k,l,m) \in A}. P \ i \ j \ k \ l \ m) \text{ is } H$

**proof** –

**have**  $(\bigcap_{(i,j,k,l,m) \in A}. P \ i \ j \ k \ l \ m)$   
 $= (\bigcap_{x \in A}. P \ (\text{fst } x) \ (\text{fst } (\text{snd } x)) \ (\text{fst } (\text{snd } (\text{snd } x))) \ (\text{fst } (\text{snd } (\text{snd } (\text{snd } x)))) \ (\text{snd } (\text{snd } (\text{snd } (\text{snd } (\text{snd } x)))))$   
**by** (*rel-auto*)

**also have** ... *is*  $H$

**by** (*metis* (*mono-tags*) *UINF-mem-Continuous-closed assms(1) assms(2) assms(3) prod.collapse*)  
**finally show** *?thesis* .

qed

**lemma** *UINF-ind-closed* [closure]:  
**assumes** *Continuous*  $H \wedge i. P\ i = \text{true} \wedge i. Q\ i \text{ is } H$   
**shows** *UINF*  $P\ Q \text{ is } H$   
**proof** –  
**from** *assms*(2) **have** *UINF*  $P\ Q = (\bigcap i. Q\ i)$   
**by** (*rel-auto*)  
**also have** ... *is*  $H$   
**using** *UINF-mem-Continuous-closed*[of  $H\ UNIV\ P$ ]  
**by** (*simp add: Sup-Continuous-closed UINF-as-Sup-collect' assms*)  
**finally show** ?thesis .  
qed

All continuous functions are also Scott-continuous

**lemma** *sup-continuous-Continuous* [closure]: *Continuous*  $F \implies \text{sup-continuous } F$   
**by** (*simp add: Continuous-def sup-continuous-def*)

**lemma** *USUP-healthy*:  $A \subseteq \llbracket H \rrbracket_H \implies (\bigsqcup P \in A. F(P)) = (\bigsqcup P \in A. F(H(P)))$   
**by** (*rule USUP-cong, simp add: Healthy-subset-member*)

**lemma** *UINF-healthy*:  $A \subseteq \llbracket H \rrbracket_H \implies (\bigcap P \in A. F(P)) = (\bigcap P \in A. F(H(P)))$   
**by** (*rule UINF-cong, simp add: Healthy-subset-member*)

end

## 15 Alphabetised Relations

**theory** *utp-rel*

**imports**

*utp-pred-laws*

*utp-healthy*

*utp-lift*

*utp-tactics*

**begin**

An alphabetised relation is simply a predicate whose state-space is a product type. In this theory we construct the core operators of the relational calculus, and prove a library of associated theorems, based on Chapters 2 and 5 of the UTP book [22].

### 15.1 Relational Alphabets

We set up convenient syntax to refer to the input and output parts of the alphabet, as is common in UTP. Since we are in a product space, these are simply the lenses  $\text{fst}_L$  and  $\text{snd}_L$ .

**definition**  $\text{in}\alpha :: ('a \implies 'a \times 'b)$  **where**  
*[lens-defs]*:  $\text{in}\alpha = \text{fst}_L$

**definition**  $\text{out}\alpha :: ('b \implies 'a \times 'b)$  **where**  
*[lens-defs]*:  $\text{out}\alpha = \text{snd}_L$

**lemma** *in $\alpha$ -uvar* [simp]: *vwb-lens*  $\text{in}\alpha$   
**by** (*unfold-locales, auto simp add: in $\alpha$ -def*)

**lemma** *out $\alpha$ -uvar* [simp]: *vwb-lens out $\alpha$*   
**by** (*unfold-locales*, *auto simp add: out $\alpha$ -def*)

**lemma** *var-in-alpha* [simp]: *x ;<sub>L</sub> in $\alpha$  = ivar x*  
**by** (*simp add: fst-lens-def in $\alpha$ -def in-var-def*)

**lemma** *var-out-alpha* [simp]: *x ;<sub>L</sub> out $\alpha$  = ovar x*  
**by** (*simp add: out $\alpha$ -def out-var-def snd-lens-def*)

**lemma** *drop-pre-inv* [simp]:  $\llbracket \text{out}\alpha \# p \rrbracket \implies \llbracket p \rrbracket_{<} = p$   
**by** (*pred-simp*)

**lemma** *usubst-lookup-ivar-unrest* [usubst]:  
*in $\alpha$  #  $\sigma \implies \langle \sigma \rangle_s (\text{ivar } x) = \$x$*   
**by** (*rel-simp*, *metis fstI*)

**lemma** *usubst-lookup-ovar-unrest* [usubst]:  
*out $\alpha$  #  $\sigma \implies \langle \sigma \rangle_s (\text{ovar } x) = \$x'$*   
**by** (*rel-simp*, *metis sndI*)

**lemma** *out-alpha-in-indep* [simp]:  
*out $\alpha$   $\bowtie$  in-var x in-var x  $\bowtie$  out $\alpha$*   
**by** (*simp-all add: in-var-def out $\alpha$ -def lens-indep-def fst-lens-def snd-lens-def lens-comp-def*)

**lemma** *in-alpha-out-indep* [simp]:  
*in $\alpha$   $\bowtie$  out-var x out-var x  $\bowtie$  in $\alpha$*   
**by** (*simp-all add: in-var-def in $\alpha$ -def lens-indep-def fst-lens-def lens-comp-def*)

The following two functions lift a predicate substitution to a relational one.

**abbreviation** *usubst-rel-lift* ::  $'\alpha \text{ usubst} \Rightarrow ('\alpha \times '\beta) \text{ usubst } (\llbracket - \rrbracket_s)$  **where**  
 $\llbracket \sigma \rrbracket_s \equiv \sigma \oplus_s \text{in}\alpha$

**abbreviation** *usubst-rel-drop* ::  $(''\alpha \times '\alpha) \text{ usubst} \Rightarrow '\alpha \text{ usubst } (\llbracket - \rrbracket_s)$  **where**  
 $\llbracket \sigma \rrbracket_s \equiv \sigma \upharpoonright_s \text{in}\alpha$

The alphabet of a relation then consists wholly of the input and output portions.

**lemma** *alpha-in-out*:  
 $\Sigma \approx_L \text{in}\alpha +_L \text{out}\alpha$   
**by** (*simp add: fst-snd-id-lens in $\alpha$ -def lens-equiv-refl out $\alpha$ -def*)

## 15.2 Relational Types and Operators

We create type synonyms for conditions (which are simply predicates) – i.e. relations without dashed variables –, alphabetised relations where the input and output alphabet can be different, and finally homogeneous relations.

**type-synonym**  $'\alpha \text{ cond} = '\alpha \text{ upred}$   
**type-synonym**  $(''\alpha, '\beta) \text{ urel} = (''\alpha \times '\beta) \text{ upred}$   
**type-synonym**  $'\alpha \text{ hrel} = (''\alpha \times '\alpha) \text{ upred}$   
**type-synonym**  $('a, '\alpha) \text{ hexpr} = ('a, '\alpha \times '\alpha) \text{ uexpr}$

### translations

$(\text{type}) (''\alpha, '\beta) \text{ urel} \leq (\text{type}) (''\alpha \times '\beta) \text{ upred}$

We set up some overloaded constants for sequential composition and the identity in case we want to overload their definitions later.

**consts**

$useq \quad :: 'a \Rightarrow 'b \Rightarrow 'c \text{ (infixr ;; 61)}$   
 $uassigns \quad :: 'a \text{ } usubst \Rightarrow 'b \text{ } (\cdot)_a$   
 $uskip \quad :: 'a \text{ } (II)$

We define a specialised version of the conditional where the condition can refer only to undashed variables, as is usually the case in programs, but not universally in UTP models. We implement this by lifting the condition predicate into the relational state-space with construction  $\lceil b \rceil_{<}$ .

**definition**  $lift-rcond \text{ } (\lceil \cdot \rceil_{<})$  **where**

$[upred-defs]: \lceil b \rceil_{<} = \lceil b \rceil_{<}$

**abbreviation**

$rcond \quad :: ('a, 'b) \text{ } urel \Rightarrow 'a \text{ } cond \Rightarrow ('a, 'b) \text{ } urel \Rightarrow ('a, 'b) \text{ } urel$   
 $((\beta \cdot \triangleleft \cdot \triangleright_r / \cdot) [52, 0, 53] \ 52)$   
**where**  $(P \triangleleft b \triangleright_r Q) \equiv (P \triangleleft \lceil b \rceil_{<} \triangleright Q)$

Sequential composition is heterogeneous, and simply requires that the output alphabet of the first matches then input alphabet of the second. We define it by lifting HOL's built-in relational composition operator  $((O))$ . Since this returns a set, the definition states that the state binding  $b$  is an element of this set.

**lift-definition**  $seqr :: ('a, 'b) \text{ } urel \Rightarrow ('b, 'c) \text{ } urel \Rightarrow ('a \times 'c) \text{ } upred$   
**is**  $\lambda P Q b. b \in (\{p. P p\} O \{q. Q q\})$ .

**ad hoc-overloading**

$useq \text{ } seqr$

We also set up a homogeneous sequential composition operator, and versions of *true* and *false* that are explicitly typed by a homogeneous alphabet.

**abbreviation**  $seqh \quad :: 'a \text{ } hrel \Rightarrow 'a \text{ } hrel \Rightarrow 'a \text{ } hrel \text{ (infixr ;;}_h \text{ 61)}$  **where**  
 $seqh \ P \ Q \equiv (P ;; Q)$

**abbreviation**  $truer \quad :: 'a \text{ } hrel \text{ } (true_h)$  **where**

$truer \equiv true$

**abbreviation**  $falserr \quad :: 'a \text{ } hrel \text{ } (false_h)$  **where**

$falserr \equiv false$

We define the relational converse operator as an alphabet extrusion on the bijective lens  $swap_L$  that swaps the elements of the product state-space.

**abbreviation**  $conv-r \quad :: ('a, 'a \times 'b) \text{ } uexpr \Rightarrow ('a, 'b \times 'a) \text{ } uexpr \text{ } (- [999] \ 999)$   
**where**  $conv-r \ e \equiv e \oplus_p swap_L$

Assignment is defined using substitutions, where latter defines what each variable should map to. This approach, which is originally due to Back [3], permits more general assignment expressions. The definition of the operator identifies the after state binding,  $b'$ , with the substitution function applied to the before state binding  $b$ .

**lift-definition**  $assigns-r \quad :: 'a \text{ } usubst \Rightarrow 'a \text{ } hrel$

**is**  $\lambda \sigma \ (b, b'). \ b' = \sigma(b)$ .

**ad hoc-overloading**

$uassigns \text{ } assigns-r$

Relational identity, or skip, is then simply an assignment with the identity substitution: it simply identifies all variables.

**definition**  $skip-r :: 'a \text{ hrel} \text{ where}$

$[urel-defs]: skip-r = assigns-r \text{ id}$

**ad hoc-overloading**

$uskip \ skip-r$

Non-deterministic assignment, also known as “choose”, assigns an arbitrarily chosen value to the given variable

**definition**  $nd-assign :: ('a \Rightarrow 'a) \Rightarrow 'a \text{ hrel} \text{ where}$

$[urel-defs]: nd-assign \ x = (\prod \ v \cdot assigns-r \ [x \mapsto_s \ll v \gg])$

We set up iterated sequential composition which iterates an indexed predicate over the elements of a list.

**definition**  $seqr-iter :: 'a \text{ list} \Rightarrow ('a \Rightarrow 'b \text{ hrel}) \Rightarrow 'b \text{ hrel} \text{ where}$

$[urel-defs]: seqr-iter \ xs \ P = foldr \ (\lambda \ i \ Q. P(i) ;; Q) \ xs \ II$

A singleton assignment simply applies a singleton substitution function, and similarly for a double assignment.

**abbreviation**  $assign-r :: ('t \Rightarrow 'a) \Rightarrow ('t, 'a) \text{ uexpr} \Rightarrow 'a \text{ hrel}$

**where**  $assign-r \ x \ v \equiv \langle [x \mapsto_s v] \rangle_a$

**abbreviation**  $assign-2-r ::$

$('t1 \Rightarrow 'a) \Rightarrow ('t2 \Rightarrow 'a) \Rightarrow ('t1, 'a) \text{ uexpr} \Rightarrow ('t2, 'a) \text{ uexpr} \Rightarrow 'a \text{ hrel}$

**where**  $assign-2-r \ x \ y \ u \ v \equiv assigns-r \ [x \mapsto_s u, y \mapsto_s v]$

We also define the alphabetised skip operator that identifies all input and output variables in the given alphabet lens. All other variables are unrestricted. We also set up syntax for it.

**definition**  $skip-ra :: ('b, 'a) \text{ lens} \Rightarrow 'a \text{ hrel} \text{ where}$

$[urel-defs]: skip-ra \ v = (\$v' =_u \$v)$

Similarly, we define the alphabetised assignment operator.

**definition**  $assigns-ra :: 'a \text{ usubst} \Rightarrow ('b, 'a) \text{ lens} \Rightarrow 'a \text{ hrel} \ (\langle - \rangle_-) \text{ where}$

$\langle \sigma \rangle_a = (\lceil \sigma \rceil_s \uparrow skip-ra \ a)$

Assumptions ( $c^\top$ ) and assertions ( $c_\perp$ ) are encoded as conditionals. An assumption behaves like skip if the condition is true, and otherwise behaves like *false* (miracle). An assertion is the same, but yields *true*, which is an abort. They are the same as tests, as in Kleene Algebra with Tests [24, 1] (KAT), which embeds a Boolean algebra into a Kleene algebra to represent conditions.

**definition**  $rassume :: 'a \text{ upred} \Rightarrow 'a \text{ hrel} \text{ where}$

$[urel-defs]: rassume \ c = II \triangleleft c \triangleright_r \text{ false}$

**definition**  $raassert :: 'a \text{ upred} \Rightarrow 'a \text{ hrel} \text{ where}$

$[urel-defs]: raassert \ c = II \triangleleft c \triangleright_r \text{ true}$

We define two variants of while loops based on strongest and weakest fixed points. The former is *false* for an infinite loop, and the latter is *true*.

**definition**  $while-top :: 'a \text{ cond} \Rightarrow 'a \text{ hrel} \Rightarrow 'a \text{ hrel} \text{ where}$

$[urel-defs]: while-top \ b \ P = (\nu \ X \cdot (P ;; X) \triangleleft b \triangleright_r II)$

**definition**  $while-bot :: 'a \text{ cond} \Rightarrow 'a \text{ hrel} \Rightarrow 'a \text{ hrel} \text{ where}$

$[urel-defs]: while-bot \ b \ P = (\mu \ X \cdot (P ;; X) \triangleleft b \triangleright_r II)$

While loops with invariant decoration (cf. [1]) – partial correctness.

**definition**  $\text{while-inv} :: 'a \text{ cond} \Rightarrow 'a \text{ cond} \Rightarrow 'a \text{ hrel} \Rightarrow 'a \text{ hrel}$  **where**  
 $[\text{urel-defs}]: \text{while-inv } b \text{ } p \text{ } S = \text{while-top } b \text{ } S$

While loops with invariant decoration – total correctness.

**definition**  $\text{while-inv-bot} :: 'a \text{ cond} \Rightarrow 'a \text{ cond} \Rightarrow 'a \text{ hrel} \Rightarrow 'a \text{ hrel}$  **where**  
 $[\text{urel-defs}]: \text{while-inv-bot } b \text{ } p \text{ } S = \text{while-bot } b \text{ } S$

While loops with invariant and variant decorations – total correctness.

**definition**  $\text{while-vrt} ::$   
 $'a \text{ cond} \Rightarrow 'a \text{ cond} \Rightarrow (\text{nat}, 'a) \text{ uexpr} \Rightarrow 'a \text{ hrel} \Rightarrow 'a \text{ hrel}$  **where**  
 $[\text{urel-defs}]: \text{while-vrt } b \text{ } p \text{ } v \text{ } S = \text{while-bot } b \text{ } S$

**syntax**

$\text{-uassume} \quad :: \text{uexp} \Rightarrow \text{logic } ([\cdot]^\top)$   
 $\text{-uassume} \quad :: \text{uexp} \Rightarrow \text{logic } (?[\cdot])$   
 $\text{-uassert} \quad :: \text{uexp} \Rightarrow \text{logic } (\{\cdot\}_\perp)$   
 $\text{-uwhile} \quad :: \text{uexp} \Rightarrow \text{logic} \Rightarrow \text{logic } (\text{while}^\top - \text{do} - \text{od})$   
 $\text{-uwhile-top} \quad :: \text{uexp} \Rightarrow \text{logic} \Rightarrow \text{logic } (\text{while} - \text{do} - \text{od})$   
 $\text{-uwhile-bot} \quad :: \text{uexp} \Rightarrow \text{logic} \Rightarrow \text{logic } (\text{while}_\perp - \text{do} - \text{od})$   
 $\text{-uwhile-inv} \quad :: \text{uexp} \Rightarrow \text{uexp} \Rightarrow \text{logic} \Rightarrow \text{logic } (\text{while} - \text{invr} - \text{do} - \text{od})$   
 $\text{-uwhile-inv-bot} \quad :: \text{uexp} \Rightarrow \text{uexp} \Rightarrow \text{logic} \Rightarrow \text{logic } (\text{while}_\perp - \text{invr} - \text{do} - \text{od } 71)$   
 $\text{-uwhile-vrt} \quad :: \text{uexp} \Rightarrow \text{uexp} \Rightarrow \text{uexp} \Rightarrow \text{logic} \Rightarrow \text{logic } (\text{while} - \text{invr} - \text{vrt} - \text{do} - \text{od})$

**translations**

$\text{-uassume } b == \text{CONST } \text{rassume } b$   
 $\text{-uassert } b == \text{CONST } \text{rassert } b$   
 $\text{-uwhile } b \text{ } P == \text{CONST } \text{while-top } b \text{ } P$   
 $\text{-uwhile-top } b \text{ } P == \text{CONST } \text{while-top } b \text{ } P$   
 $\text{-uwhile-bot } b \text{ } P == \text{CONST } \text{while-bot } b \text{ } P$   
 $\text{-uwhile-inv } b \text{ } p \text{ } S == \text{CONST } \text{while-inv } b \text{ } p \text{ } S$   
 $\text{-uwhile-inv-bot } b \text{ } p \text{ } S == \text{CONST } \text{while-inv-bot } b \text{ } p \text{ } S$   
 $\text{-uwhile-vrt } b \text{ } p \text{ } v \text{ } S == \text{CONST } \text{while-vrt } b \text{ } p \text{ } v \text{ } S$

We implement a poor man's version of alphabet restriction that hides a variable within a relation.

**definition**  $\text{rel-var-res} :: 'a \text{ hrel} \Rightarrow ('a \Longrightarrow 'a) \Rightarrow 'a \text{ hrel}$  (**infix**  $\downarrow_\alpha$  80) **where**  
 $[\text{urel-defs}]: P \downarrow_\alpha x = (\exists \$x \cdot \exists \$x' \cdot P)$

Alphabet extension and restriction add additional variables by the given lens in both their primed and unprimed versions.

**definition**  $\text{rel-aext} :: 'b \text{ hrel} \Rightarrow ('b \Longrightarrow 'a) \Rightarrow 'a \text{ hrel}$   
**where**  $[\text{upred-defs}]: \text{rel-aext } P \text{ } a = P \oplus_p (a \times_L a)$

**definition**  $\text{rel-ares} :: 'a \text{ hrel} \Rightarrow ('b \Longrightarrow 'a) \Rightarrow 'b \text{ hrel}$   
**where**  $[\text{upred-defs}]: \text{rel-ares } P \text{ } a = (P \downarrow_p (a \times a))$

We next describe frames and antiframes with the help of lenses. A frame states that  $P$  defines how variables in  $a$  changed, and all those outside of  $a$  remain the same. An antiframe describes the converse: all variables outside  $a$  are specified by  $P$ , and all those in remain the same. For more information please see [25].

**definition**  $\text{frame} :: ('a \Longrightarrow 'a) \Rightarrow 'a \text{ hrel} \Rightarrow 'a \text{ hrel}$  **where**  
 $[\text{urel-defs}]: \text{frame } a \text{ } P = (P \wedge \$\mathbf{v}' =_u \$\mathbf{v} \oplus \$\mathbf{v}' \text{ on } \&a)$

**definition** *antiframe* :: ( $'a \Rightarrow 'α \Rightarrow 'α \text{ hrel} \Rightarrow 'α \text{ hrel}$  **where**  
 $[urel-defs]: \text{antiframe } a \ P = (P \wedge \$\mathbf{v}' =_u \$\mathbf{v}' \oplus \$\mathbf{v} \text{ on } \&a)$

Frame extension combines alphabet extension with the frame operator to both add additional variables and then frame those.

**definition** *rel-frex* :: ( $'β \Rightarrow 'α \Rightarrow 'β \text{ hrel} \Rightarrow 'α \text{ hrel}$  **where**  
 $[upred-defs]: \text{rel-frex } a \ P = \text{frame } a \ (\text{rel-aext } P \ a)$

The nameset operator can be used to hide a portion of the after-state that lies outside the lens  $a$ . It can be useful to partition a relation's variables in order to conjoin it with another relation.

**definition** *nameset* :: ( $'a \Rightarrow 'α \Rightarrow 'α \text{ hrel} \Rightarrow 'α \text{ hrel}$  **where**  
 $[urel-defs]: \text{nameset } a \ P = (P \upharpoonright_v \{\$ \mathbf{v}, \$a'\})$

### 15.3 Syntax Translations

#### syntax

- Alternative traditional conditional syntax
- utp-if ::  $uexp \Rightarrow logic \Rightarrow logic \Rightarrow logic \ ((\text{if}_u \ (-) / \text{then } (-) / \text{else } (-)) \ [0, 0, 71] \ 71)$
- Iterated sequential composition
- segr-iter ::  $pttrn \Rightarrow 'a \text{ list} \Rightarrow 'σ \text{ hrel} \Rightarrow 'σ \text{ hrel} \ ((3;; - : - \cdot / -) \ [0, 0, 10] \ 10)$
- Single and multiple assignement
- assignment ::  $svids \Rightarrow uexprs \Rightarrow 'α \text{ hrel} \ ((-') := '(-))$
- assignment ::  $svids \Rightarrow uexprs \Rightarrow 'α \text{ hrel} \ (\text{infixr} := 62)$
- Non-deterministic assignment
- nd-assign ::  $svids \Rightarrow logic \ (- := * \ [62] \ 62)$
- Substitution constructor
- mk-usubst ::  $svids \Rightarrow uexprs \Rightarrow 'α \text{ usubst}$
- Alphabetised skip
- skip-ra ::  $salpha \Rightarrow logic \ (II.)$
- Frame
- frame ::  $salpha \Rightarrow logic \Rightarrow logic \ (-:[-] \ [99,0] \ 100)$
- Antiframe
- antiframe ::  $salpha \Rightarrow logic \Rightarrow logic \ (-:[-] \ [79,0] \ 80)$
- Relational Alphabet Extension
- rel-aext ::  $logic \Rightarrow salpha \Rightarrow logic \ (\text{infixl } \oplus_r \ 90)$
- Relational Alphabet Restriction
- rel-ares ::  $logic \Rightarrow salpha \Rightarrow logic \ (\text{infixl } \upharpoonright_r \ 90)$
- Frame Extension
- rel-frex ::  $salpha \Rightarrow logic \Rightarrow logic \ (-:[-]^+ \ [99,0] \ 100)$
- Nameset
- nameset ::  $salpha \Rightarrow logic \Rightarrow logic \ (ns - \cdot - \ [0,999] \ 999)$

#### translations

- utp-if  $b \ P \ Q \Rightarrow P \triangleleft b \triangleright_r Q$
- ;;  $x : l \cdot P \Rightarrow (CONST \text{ segr-iter}) \ l \ (\lambda x. P)$
- mk-usubst  $\sigma \ (-\text{svid-unit } x) \ v \Rightarrow \sigma(\&x \mapsto_s v)$
- mk-usubst  $\sigma \ (-\text{svid-list } x \ xs) \ (-uexprs \ v \ vs) \Rightarrow (-\text{mk-usubst } (\sigma(\&x \mapsto_s v)) \ xs \ vs)$
- assignment  $xs \ vs \Rightarrow CONST \ uassigns \ (-\text{mk-usubst } (CONST \text{ id}) \ xs \ vs)$
- assignment  $x \ v \leq CONST \ uassigns \ (CONST \text{ subst-upd } (CONST \text{ id}) \ x \ v)$
- assignment  $x \ v \leq -\text{assignment } (-\text{spvar } x) \ v$
- nd-assign  $x \Rightarrow CONST \ nd\text{-assign } (-\text{mk-svid-list } x)$
- nd-assign  $x \leq CONST \ nd\text{-assign } x$
- $x, y := u, v \leq CONST \ uassigns \ (CONST \text{ subst-upd } (CONST \text{ subst-upd } (CONST \text{ id}) \ (CONST \text{ svar } u) \ (CONST \text{ svar } y) \ v)$

```

-skip-ra v  $\Rightarrow$  CONST skip-ra v
-frame x P  $\Rightarrow$  CONST frame x P
-frame (-salphaset (-salphamk x)) P  $\leq$  CONST frame x P
-antiframe x P  $\Rightarrow$  CONST antiframe x P
-antiframe (-salphaset (-salphamk x)) P  $\leq$  CONST antiframe x P
-nameset x P  $\equiv$  CONST nameset x P
-rel-aext P a  $\equiv$  CONST rel-aext P a
-rel-ares P a  $\equiv$  CONST rel-ares P a
-rel-frext a P  $\equiv$  CONST rel-frext a P

```

The following code sets up pretty-printing for homogeneous relational expressions. We cannot do this via the “translations” command as we only want the rule to apply when the input and output alphabet types are the same. The code has to deconstruct a  $(\alpha, \alpha')$  *uexpr* type, determine that it is relational (product alphabet), and then checks if the types *alpha* and *beta* are the same. If they are, the type is printed as a *hexpr*. Otherwise, we have no match. We then set up a regular translation for the *hrel* type that uses this.

```

print-translation <
let
fun tr' ctxt [ a
, Const (@{type-syntax prod},-) $ alpha $ beta ] =
  if (alpha = beta)
  then Syntax.const @{type-syntax hexpr} $ a $ alpha
  else raise Match;
in [(@{type-syntax uexpr},tr')]
end
>

```

#### translations

```
(type) ' $\alpha$  hrel  $\leq$  (type) (bool, ' $\alpha$ ) hexpr
```

### 15.4 Relation Properties

We describe some properties of relations, including functional and injective relations. We also provide operators for extracting the domain and range of a UTP relation.

**definition** *ufunctional* ::  $(\alpha, \alpha') \text{ urel} \Rightarrow \text{bool}$   
**where** [*urel-defs*]: *ufunctional* *R*  $\longleftrightarrow II \sqsubseteq R^-$  ;; *R*

**definition** *uinj* ::  $(\alpha, \alpha') \text{ urel} \Rightarrow \text{bool}$   
**where** [*urel-defs*]: *uinj* *R*  $\longleftrightarrow II \sqsubseteq R$  ;; *R*<sup>-</sup>

**definition** *Dom* ::  $\alpha' \text{ hrel} \Rightarrow \alpha \text{ upred}$   
**where** [*upred-defs*]: *Dom* *P* =  $[\exists \mathbf{v}' \cdot P]_{<}$

**definition** *Ran* ::  $\alpha' \text{ hrel} \Rightarrow \alpha \text{ upred}$   
**where** [*upred-defs*]: *Ran* *P* =  $[\exists \mathbf{v} \cdot P]_{>}$

— Configuration for UTP tactics.

**update-uexpr-rep-eq-thms** — Reread *rep-eq* theorems.

### 15.5 Introduction laws

**lemma** *urel-refine-ext*:  

$$[\bigwedge s s'. P[\llbracket s \rrbracket, \llbracket s' \rrbracket / \mathbf{v}, \mathbf{v}']] \sqsubseteq Q[\llbracket s \rrbracket, \llbracket s' \rrbracket / \mathbf{v}, \mathbf{v}']] \implies P \sqsubseteq Q$$



by (rel-auto)

**lemma** *urel-eq-ext*:

$\llbracket \bigwedge s s'. P[\llbracket s \gg, \ll s' \gg / \$v, \$v' \rrbracket = Q[\llbracket s \gg, \ll s' \gg / \$v, \$v' \rrbracket] \rrbracket \implies P = Q$   
by (rel-auto)

## 15.6 Unrestriction Laws

**lemma** *unrest-iuvar* [unrest]:  $out\alpha \# \$x$

by (metis fst-snd-lens-indep lift-pre-var out $\alpha$ -def unrest-aext-indep)

**lemma** *unrest-ouvar* [unrest]:  $in\alpha \# \$x'$

by (metis in $\alpha$ -def lift-post-var snd-fst-lens-indep unrest-aext-indep)

**lemma** *unrest-semir-undash* [unrest]:

fixes  $x :: ('a \implies 'a)$

assumes  $\$x \# P$

shows  $\$x \# P ;; Q$

using *assms* by (rel-auto)

**lemma** *unrest-semir-dash* [unrest]:

fixes  $x :: ('a \implies 'a)$

assumes  $\$x' \# Q$

shows  $\$x' \# P ;; Q$

using *assms* by (rel-auto)

**lemma** *unrest-cond* [unrest]:

$\llbracket x \# P; x \# b; x \# Q \rrbracket \implies x \# P \triangleleft b \triangleright Q$

by (rel-auto)

**lemma** *unrest-lift-rcond* [unrest]:

$x \# [b]_< \implies x \# [b]_{\leftarrow}$

by (simp add: lift-rcond-def)

**lemma** *unrest-in $\alpha$ -var* [unrest]:

$\llbracket mwb\text{-}lens\ x; in\alpha \# (P :: ('a, ('a \times 'b))\ uexpr) \rrbracket \implies \$x \# P$

by (rel-auto)

**lemma** *unrest-out $\alpha$ -var* [unrest]:

$\llbracket mwb\text{-}lens\ x; out\alpha \# (P :: ('a, ('a \times 'b))\ uexpr) \rrbracket \implies \$x' \# P$

by (rel-auto)

**lemma** *unrest-pre-out $\alpha$*  [unrest]:  $out\alpha \# [b]_<$

by (transfer, auto simp add: out $\alpha$ -def)

**lemma** *unrest-post-in $\alpha$*  [unrest]:  $in\alpha \# [b]_>$

by (transfer, auto simp add: in $\alpha$ -def)

**lemma** *unrest-pre-in-var* [unrest]:

$x \# p1 \implies \$x \# [p1]_<$

by (transfer, simp)

**lemma** *unrest-post-out-var* [unrest]:

$x \# p1 \implies \$x' \# [p1]_>$

by (transfer, simp)

**lemma** *unrest-convr-outα* [*unrest*]:  
 $in\alpha \# p \implies out\alpha \# p^-$   
**by** (*transfer*, *auto simp add: lens-defs*)

**lemma** *unrest-convr-inα* [*unrest*]:  
 $out\alpha \# p \implies in\alpha \# p^-$   
**by** (*transfer*, *auto simp add: lens-defs*)

**lemma** *unrest-in-rel-var-res* [*unrest*]:  
 $vwb\text{-}lens\ x \implies \$x \# (P \upharpoonright_\alpha x)$   
**by** (*simp add: rel-var-res-def unrest*)

**lemma** *unrest-out-rel-var-res* [*unrest*]:  
 $vwb\text{-}lens\ x \implies \$x' \# (P \upharpoonright_\alpha x)$   
**by** (*simp add: rel-var-res-def unrest*)

**lemma** *unrest-out-alpha-usubst-rel-lift* [*unrest*]:  
 $out\alpha \# [\sigma]_s$   
**by** (*rel-auto*)

**lemma** *unrest-in-rel-aext* [*unrest*]:  $x \bowtie y \implies \$y \# P \oplus_r x$   
**by** (*simp add: rel-aext-def unrest-aext-indep*)

**lemma** *unrest-out-rel-aext* [*unrest*]:  $x \bowtie y \implies \$y' \# P \oplus_r x$   
**by** (*simp add: rel-aext-def unrest-aext-indep*)

**lemma** *rel-aext-false* [*alpha*]:  
 $false \oplus_r a = false$   
**by** (*pred-auto*)

**lemma** *rel-aext-seq* [*alpha*]:  
 $weak\text{-}lens\ a \implies (P ;; Q) \oplus_r a = (P \oplus_r a ;; Q \oplus_r a)$   
**apply** (*rel-auto*)  
**apply** (*rename-tac aa b y*)  
**apply** (*rule-tac x=create<sub>a</sub> y in exI*)  
**apply** (*simp*)  
**done**

**lemma** *rel-aext-cond* [*alpha*]:  
 $(P \triangleleft b \triangleright_r Q) \oplus_r a = (P \oplus_r a \triangleleft b \oplus_p a \triangleright_r Q \oplus_r a)$   
**by** (*rel-auto*)

## 15.7 Substitution laws

**lemma** *subst-seq-left* [*usubst*]:  
 $out\alpha \# \sigma \implies \sigma \dagger (P ;; Q) = (\sigma \dagger P) ;; Q$   
**by** (*rel-simp*, (*metis (no-types, lifting) Pair-inject surjective-pairing*)+)

**lemma** *subst-seq-right* [*usubst*]:  
 $in\alpha \# \sigma \implies \sigma \dagger (P ;; Q) = P ;; (\sigma \dagger Q)$   
**by** (*rel-simp*, (*metis (no-types, lifting) Pair-inject surjective-pairing*)+)

The following laws support substitution in heterogeneous relations for polymorphically typed literal expressions. These cannot be supported more generically due to limitations in HOL's type system. The laws are presented in a slightly strange way so as to be as general as possible.

**lemma** *bool-seqr-laws* [usubst]:

**fixes**  $x :: (bool \implies 'a)$

**shows**

$\bigwedge P Q \sigma. \sigma(\$x \mapsto_s true) \dagger (P ;; Q) = \sigma \dagger (P \llbracket true/\$x \rrbracket ;; Q)$   
 $\bigwedge P Q \sigma. \sigma(\$x \mapsto_s false) \dagger (P ;; Q) = \sigma \dagger (P \llbracket false/\$x \rrbracket ;; Q)$   
 $\bigwedge P Q \sigma. \sigma(\$x' \mapsto_s true) \dagger (P ;; Q) = \sigma \dagger (P ;; Q \llbracket true/\$x' \rrbracket)$   
 $\bigwedge P Q \sigma. \sigma(\$x' \mapsto_s false) \dagger (P ;; Q) = \sigma \dagger (P ;; Q \llbracket false/\$x' \rrbracket)$   
**by** (rel-auto)+

**lemma** *zero-one-seqr-laws* [usubst]:

**fixes**  $x :: (- \implies 'a)$

**shows**

$\bigwedge P Q \sigma. \sigma(\$x \mapsto_s 0) \dagger (P ;; Q) = \sigma \dagger (P \llbracket 0/\$x \rrbracket ;; Q)$   
 $\bigwedge P Q \sigma. \sigma(\$x \mapsto_s 1) \dagger (P ;; Q) = \sigma \dagger (P \llbracket 1/\$x \rrbracket ;; Q)$   
 $\bigwedge P Q \sigma. \sigma(\$x' \mapsto_s 0) \dagger (P ;; Q) = \sigma \dagger (P ;; Q \llbracket 0/\$x' \rrbracket)$   
 $\bigwedge P Q \sigma. \sigma(\$x' \mapsto_s 1) \dagger (P ;; Q) = \sigma \dagger (P ;; Q \llbracket 1/\$x' \rrbracket)$   
**by** (rel-auto)+

**lemma** *numeral-seqr-laws* [usubst]:

**fixes**  $x :: (- \implies 'a)$

**shows**

$\bigwedge P Q \sigma. \sigma(\$x \mapsto_s numeral\ n) \dagger (P ;; Q) = \sigma \dagger (P \llbracket numeral\ n/\$x \rrbracket ;; Q)$   
 $\bigwedge P Q \sigma. \sigma(\$x' \mapsto_s numeral\ n) \dagger (P ;; Q) = \sigma \dagger (P ;; Q \llbracket numeral\ n/\$x' \rrbracket)$   
**by** (rel-auto)+

**lemma** *usubst-condr* [usubst]:

$\sigma \dagger (P \triangleleft b \triangleright Q) = (\sigma \dagger P \triangleleft \sigma \dagger b \triangleright \sigma \dagger Q)$

**by** (rel-auto)

**lemma** *subst-skip-r* [usubst]:

$out\alpha \# \sigma \implies \sigma \dagger II = \langle \llbracket \sigma \rrbracket_s \rangle_a$

**by** (rel-simp, (metis (mono-tags, lifting) prod.sel(1) sndI surjective-pairing)+)

**lemma** *subst-pre-skip* [usubst]:  $\llbracket \sigma \rrbracket_s \dagger II = \langle \sigma \rangle_a$

**by** (rel-auto)

**lemma** *subst-rel-lift-seq* [usubst]:

$\llbracket \sigma \rrbracket_s \dagger (P ;; Q) = (\llbracket \sigma \rrbracket_s \dagger P) ;; Q$

**by** (rel-auto)

**lemma** *subst-rel-lift-comp* [usubst]:

$\llbracket \sigma \rrbracket_s \circ \llbracket \varrho \rrbracket_s = \llbracket \sigma \circ \varrho \rrbracket_s$

**by** (rel-auto)

**lemma** *usubst-upd-in-comp* [usubst]:

$\sigma(\&in\alpha:x \mapsto_s v) = \sigma(\$x \mapsto_s v)$

**by** (simp add: pr-var-def fst-lens-def in $\alpha$ -def in-var-def)

**lemma** *usubst-upd-out-comp* [usubst]:

$\sigma(\&out\alpha:x \mapsto_s v) = \sigma(\$x' \mapsto_s v)$

**by** (simp add: pr-var-def out $\alpha$ -def out-var-def snd-lens-def)

**lemma** *subst-lift-upd* [alpha]:

**fixes**  $x :: ('a \implies 'a)$

**shows**  $\llbracket \sigma(x \mapsto_s v) \rrbracket_s = \llbracket \sigma \rrbracket_s(\$x \mapsto_s \llbracket v \rrbracket_<)$

by (simp add: alpha usubst, simp add: pr-var-def fst-lens-def in $\alpha$ -def in-var-def)

**lemma** subst-drop-upd [alpha]:

fixes  $x :: ('a \Rightarrow 'a)$

shows  $\lfloor \sigma(\$x \mapsto_s v) \rfloor_s = \lfloor \sigma \rfloor_s (x \mapsto_s \lfloor v \rfloor_<)$

by pred-simp

**lemma** subst-lift-pre [usubst]:  $\lfloor \sigma \rfloor_s \dagger \lfloor b \rfloor_< = \lfloor \sigma \dagger b \rfloor_<$

by (metis apply-subst-ext fst-vwb-lens in $\alpha$ -def)

**lemma** unrest-usubst-lift-in [unrest]:

$x \# P \Rightarrow \$x \# \lfloor P \rfloor_s$

by pred-simp

**lemma** unrest-usubst-lift-out [unrest]:

fixes  $x :: ('a \Rightarrow 'a)$

shows  $\$x' \# \lfloor P \rfloor_s$

by pred-simp

**lemma** subst-lift-cond [usubst]:  $\lfloor \sigma \rfloor_s \dagger \lfloor s \rfloor_{\leftarrow} = \lfloor \sigma \dagger s \rfloor_{\leftarrow}$

by (rel-auto)

**lemma** msubst-seq [usubst]:  $(P(x) ;; Q(x)) \llbracket x \rightarrow \ll v \gg \rrbracket = ((P(x)) \llbracket x \rightarrow \ll v \gg \rrbracket ;; (Q(x)) \llbracket x \rightarrow \ll v \gg \rrbracket)$

by (rel-auto)

## 15.8 Alphabet laws

**lemma** aext-cond [alpha]:

$(P \triangleleft b \triangleright Q) \oplus_p a = ((P \oplus_p a) \triangleleft (b \oplus_p a) \triangleright (Q \oplus_p a))$

by (rel-auto)

**lemma** aext-seq [alpha]:

$wb\text{-}lens\ a \Rightarrow ((P ;; Q) \oplus_p (a \times_L a)) = ((P \oplus_p (a \times_L a)) ;; (Q \oplus_p (a \times_L a)))$

by (rel-simp, metis wb-lens-weak weak-lens.put-get)

**lemma** rcond-lift-true [simp]:

$\lfloor true \rfloor_{\leftarrow} = true$

by rel-auto

**lemma** rcond-lift-false [simp]:

$\lfloor false \rfloor_{\leftarrow} = false$

by rel-auto

**lemma** rel-ares-aext [alpha]:

$vwb\text{-}lens\ a \Rightarrow (P \oplus_r a) \upharpoonright_r a = P$

by (rel-auto)

**lemma** rel-aext-ares [alpha]:

$\{\$a, \$a'\} \Vdash P \Rightarrow P \upharpoonright_r a \oplus_r a = P$

by (rel-auto)

**lemma** rel-aext-uses [unrest]:

$vwb\text{-}lens\ a \Rightarrow \{\$a, \$a'\} \Vdash (P \oplus_r a)$

by (rel-auto)

## 15.9 Relational unrestriction

Relational unrestriction states that a variable is both unchanged by a relation, and is not "read" by the relation.

**definition**  $RID :: ('a \implies 'α) \Rightarrow 'α \text{ hrel} \Rightarrow 'α \text{ hrel}$   
**where**  $RID\ x\ P = ((\exists\ \$x \cdot \exists\ \$x' \cdot P) \wedge \$x' =_u \$x)$

**declare**  $RID\text{-}def\ [\text{urel-defs}]$

**lemma**  $RID1: vwb\text{-}lens\ x \implies (\forall\ v. x := \ll v \gg ;; P = P ;; x := \ll v \gg) \implies RID(x)(P) = P$   
**apply**  $(rel\text{-}auto)$   
**apply**  $(metis\ vwb\text{-}lens.put\text{-}eq)$   
**apply**  $(metis\ vwb\text{-}lens\text{-}wb\ wb\text{-}lens.get\text{-}put\ wb\text{-}lens\text{-}weak\ weak\text{-}lens.put\text{-}get)$   
**done**

**lemma**  $RID2: vwb\text{-}lens\ x \implies x := \ll v \gg ;; RID(x)(P) = RID(x)(P) ;; x := \ll v \gg$   
**apply**  $(rel\text{-}auto)$   
**apply**  $(metis\ mwb\text{-}lens.put\text{-}put\ vwb\text{-}lens\text{-}mwb\ vwb\text{-}lens\text{-}wb\ wb\text{-}lens.get\text{-}put\ wb\text{-}lens\text{-}def\ weak\text{-}lens.put\text{-}get)$   
**apply**  $blast$   
**done**

**lemma**  $RID\text{-}assign\text{-}commute:$   
 $vwb\text{-}lens\ x \implies P = RID(x)(P) \longleftrightarrow (\forall\ v. x := \ll v \gg ;; P = P ;; x := \ll v \gg)$   
**by**  $(metis\ RID1\ RID2)$

**lemma**  $RID\text{-}idem:$   
 $mwb\text{-}lens\ x \implies RID(x)(RID(x)(P)) = RID(x)(P)$   
**by**  $(rel\text{-}auto)$

**lemma**  $RID\text{-}mono:$   
 $P \sqsubseteq Q \implies RID(x)(P) \sqsubseteq RID(x)(Q)$   
**by**  $(rel\text{-}auto)$

**lemma**  $RID\text{-}pr\text{-}var\ [simp]:$   
 $RID\ (pr\text{-}var\ x) = RID\ x$   
**by**  $(simp\ add: pr\text{-}var\text{-}def)$

**lemma**  $RID\text{-}skip\text{-}r:$   
 $vwb\text{-}lens\ x \implies RID(x)(II) = II$   
**apply**  $(rel\text{-}auto)$  **using**  $vwb\text{-}lens.put\text{-}eq$  **by**  $fastforce$

**lemma**  $skip\text{-}r\text{-}RID\ [closure]: vwb\text{-}lens\ x \implies II\ is\ RID(x)$   
**by**  $(simp\ add: Healthy\text{-}def\ RID\text{-}skip\text{-}r)$

**lemma**  $RID\text{-}disj:$   
 $RID(x)(P \vee Q) = (RID(x)(P) \vee RID(x)(Q))$   
**by**  $(rel\text{-}auto)$

**lemma**  $disj\text{-}RID\ [closure]: \ll P\ is\ RID(x); Q\ is\ RID(x) \rr \implies (P \vee Q)\ is\ RID(x)$   
**by**  $(simp\ add: Healthy\text{-}def\ RID\text{-}disj)$

**lemma**  $RID\text{-}conj:$   
 $vwb\text{-}lens\ x \implies RID(x)(RID(x)(P) \wedge RID(x)(Q)) = (RID(x)(P) \wedge RID(x)(Q))$   
**by**  $(rel\text{-}auto)$

**lemma** *conj-RID* [closure]:  $\llbracket \text{vwb-lens } x; P \text{ is RID}(x); Q \text{ is RID}(x) \rrbracket \implies (P \wedge Q) \text{ is RID}(x)$   
 by (metis Healthy-if Healthy-intro RID-conj)

**lemma** *RID-assigns-r-diff*:  
 $\llbracket \text{vwb-lens } x; x \# \sigma \rrbracket \implies \text{RID}(x)(\langle \sigma \rangle_a) = \langle \sigma \rangle_a$   
 apply (rel-auto)  
 apply (metis vwb-lens.put-eq)  
 apply (metis vwb-lens-wb wb-lens.get-put wb-lens-weak weak-lens.put-get)  
 done

**lemma** *assigns-r-RID* [closure]:  $\llbracket \text{vwb-lens } x; x \# \sigma \rrbracket \implies \langle \sigma \rangle_a \text{ is RID}(x)$   
 by (simp add: Healthy-def RID-assigns-r-diff)

**lemma** *RID-assign-r-same*:  
 $\text{vwb-lens } x \implies \text{RID}(x)(x := v) = \text{ID}$   
 apply (rel-auto)  
 using vwb-lens.put-eq apply fastforce  
 done

**lemma** *RID-seq-left*:  
 assumes vwb-lens  $x$   
 shows  $\text{RID}(x)(\text{RID}(x)(P) ;; Q) = (\text{RID}(x)(P) ;; \text{RID}(x)(Q))$   
**proof** –  
 have  $\text{RID}(x)(\text{RID}(x)(P) ;; Q) = ((\exists \$x \cdot \exists \$x' \cdot ((\exists \$x \cdot \exists \$x' \cdot P) \wedge \$x' =_u \$x) ;; Q) \wedge \$x' =_u \$x)$   
 by (simp add: RID-def usubst)  
 also from *assms* have  $\dots = (((\exists \$x \cdot \exists \$x' \cdot P) \wedge (\exists \$x \cdot \$x' =_u \$x)) ;; (\exists \$x' \cdot Q)) \wedge \$x' =_u \$x$   
 by (rel-auto)  
 also from *assms* have  $\dots = (((\exists \$x \cdot \exists \$x' \cdot P) ;; (\exists \$x \cdot \exists \$x' \cdot Q)) \wedge \$x' =_u \$x)$   
 apply (rel-auto)  
 apply (metis vwb-lens.put-eq)  
 apply (metis mwb-lens.put-put vwb-lens-mwb)  
 done  
 also from *assms* have  $\dots = (((\exists \$x \cdot \exists \$x' \cdot P) \wedge \$x' =_u \$x) ;; (\exists \$x \cdot \exists \$x' \cdot Q)) \wedge \$x' =_u \$x$   
 by (rel-simp, metis (full-types) mwb-lens.put-put vwb-lens-def wb-lens-weak weak-lens.put-get)  
 also have  $\dots = (((\exists \$x \cdot \exists \$x' \cdot P) \wedge \$x' =_u \$x) ;; ((\exists \$x \cdot \exists \$x' \cdot Q) \wedge \$x' =_u \$x)) \wedge \$x' =_u \$x$   
 by (rel-simp, fastforce)  
 also have  $\dots = (((\exists \$x \cdot \exists \$x' \cdot P) \wedge \$x' =_u \$x) ;; ((\exists \$x \cdot \exists \$x' \cdot Q) \wedge \$x' =_u \$x))$   
 by (rel-auto)  
 also have  $\dots = (\text{RID}(x)(P) ;; \text{RID}(x)(Q))$   
 by (rel-auto)  
 finally show ?thesis .  
**qed**

**lemma** *RID-seq-right*:  
 assumes vwb-lens  $x$   
 shows  $\text{RID}(x)(P ;; \text{RID}(x)(Q)) = (\text{RID}(x)(P) ;; \text{RID}(x)(Q))$   
**proof** –  
 have  $\text{RID}(x)(P ;; \text{RID}(x)(Q)) = ((\exists \$x \cdot \exists \$x' \cdot P ;; ((\exists \$x \cdot \exists \$x' \cdot Q) \wedge \$x' =_u \$x)) \wedge \$x' =_u \$x)$   
 by (simp add: RID-def usubst)  
 also from *assms* have  $\dots = (((\exists \$x \cdot P) ;; (\exists \$x \cdot \exists \$x' \cdot Q) \wedge (\exists \$x' \cdot \$x' =_u \$x)) \wedge \$x' =_u \$x)$   
 by (rel-auto)

by (*rel-auto*)  
 also from *assms* have ... = ((( $\exists x \cdot \exists x' \cdot P$ ) ;; ( $\exists x \cdot \exists x' \cdot Q$ ))  $\wedge x' =_u x$ )  
 apply (*rel-auto*)  
 apply (*metis vwb-lens.put-eq*)  
 apply (*metis mwb-lens.put-put vwb-lens-mwb*)  
 done  
 also from *assms* have ... = ((( $\exists x \cdot \exists x' \cdot P$ )  $\wedge x' =_u x$ ) ;; ( $\exists x \cdot \exists x' \cdot Q$ ))  $\wedge x' =_u x$ )  
 by (*rel-simp robust, metis (full-types) mwb-lens.put-put vwb-lens-def wb-lens-weak weak-lens.put-get*)  
 also have ... = ((( $\exists x \cdot \exists x' \cdot P$ )  $\wedge x' =_u x$ ) ;; (( $\exists x \cdot \exists x' \cdot Q$ )  $\wedge x' =_u x$ ))  $\wedge x' =_u x$   
 \$x\$)  
 by (*rel-simp, fastforce*)  
 also have ... = ((( $\exists x \cdot \exists x' \cdot P$ )  $\wedge x' =_u x$ ) ;; (( $\exists x \cdot \exists x' \cdot Q$ )  $\wedge x' =_u x$ ))  
 by (*rel-auto*)  
 also have ... = (*RID*( $x$ )( $P$ ) ;; *RID*( $x$ )( $Q$ ))  
 by (*rel-auto*)  
 finally show ?thesis .  
 qed

**lemma** *seqr-RID-closed* [*closure*]: [*vwb-lens*  $x$ ;  $P$  is *RID*( $x$ );  $Q$  is *RID*( $x$ )]  $\implies P$  ;;  $Q$  is *RID*( $x$ )  
 by (*metis Healthy-def RID-seq-right*)

**definition** *unrest-relation* :: ( $'a \implies 'a$ )  $\Rightarrow 'a$  *hrel*  $\Rightarrow$  bool (**infix**  $\#\#$  20)  
 where ( $x \#\# P$ )  $\longleftrightarrow$  ( $P$  is *RID*( $x$ ))

**declare** *unrest-relation-def* [*urel-defs*]

**lemma** *runrest-assign-commute*:  
 [*vwb-lens*  $x$ ;  $x \#\# P$ ]  $\implies x := \langle v \rangle$  ;;  $P = P$  ;;  $x := \langle v \rangle$   
 by (*metis RID2 Healthy-def unrest-relation-def*)

**lemma** *runrest-ident-var*:  
 assumes  $x \#\# P$   
 shows ( $x \wedge P$ ) = ( $P \wedge x'$ )  
**proof** –  
 have  $P = (x' =_u x \wedge P)$   
 by (*metis RID-def assms Healthy-def unrest-relation-def utp-pred-laws.inf.cobounded2 utp-pred-laws.inf-absorb2*)  
 moreover have ( $x' =_u x \wedge (x \wedge P)$ ) = ( $x' =_u x \wedge (P \wedge x')$ )  
 by (*rel-auto*)  
 ultimately show ?thesis  
 by (*metis utp-pred-laws.inf.assoc utp-pred-laws.inf-left-commute*)  
 qed

**lemma** *skip-r-runrest* [*unrest*]:  
*vwb-lens*  $x \implies x \#\# I$   
 by (*simp add: unrest-relation-def closure*)

**lemma** *assigns-r-runrest*:  
 [*vwb-lens*  $x$ ;  $x \# \sigma$ ]  $\implies x \#\# \langle \sigma \rangle_a$   
 by (*simp add: unrest-relation-def closure*)

**lemma** *seq-r-runrest* [*unrest*]:  
 assumes *vwb-lens*  $x$   $x \#\# P$   $x \#\# Q$   
 shows  $x \#\# (P ;; Q)$   
 using *assms* by (*simp add: unrest-relation-def closure*)

```

lemma false-runrest [unrest]:  $x \# \# \text{false}$ 
  by (rel-auto)

lemma and-runrest [unrest]:  $\llbracket x \# \# P; x \# \# Q \rrbracket \implies x \# \# (P \wedge Q)$ 
  by (metis RID-conj Healthy-def unrest-relation-def)

lemma or-runrest [unrest]:  $\llbracket x \# \# P; x \# \# Q \rrbracket \implies x \# \# (P \vee Q)$ 
  by (simp add: RID-disj Healthy-def unrest-relation-def)

end

```

## 16 Fixed-points and Recursion

```

theory utp-recursion
imports
  utp-pred-laws
  utp-rel
begin

```

### 16.1 Fixed-point Laws

```

lemma mu-id:  $(\mu X \cdot X) = \text{true}$ 
  by (simp add: antisym gfp-upperbound)

lemma mu-const:  $(\mu X \cdot P) = P$ 
  by (simp add: gfp-const)

lemma nu-id:  $(\nu X \cdot X) = \text{false}$ 
  by (meson lfp-lowerbound utp-pred-laws.bot.extremum-unique)

lemma nu-const:  $(\nu X \cdot P) = P$ 
  by (simp add: lfp-const)

lemma mu-refine-intro:
  assumes  $(C \Rightarrow S) \sqsubseteq F(C \Rightarrow S) \ (C \wedge \mu F) = (C \wedge \nu F)$ 
  shows  $(C \Rightarrow S) \sqsubseteq \mu F$ 
proof –
  from assms have  $(C \Rightarrow S) \sqsubseteq \nu F$ 
    by (simp add: lfp-lowerbound)
  with assms show ?thesis
    by (pred-auto)
qed

```

### 16.2 Obtaining Unique Fixed-points

Obtaining termination proofs via approximation chains. Theorems and proofs adapted from Chapter 2, page 63 of the UTP book [22].

```

type-synonym 'a chain = nat  $\Rightarrow$  'a upred

```

```

definition chain :: 'a chain  $\Rightarrow$  bool where
  chain Y =  $((Y\ 0 = \text{false}) \wedge (\forall i. Y\ (Suc\ i) \sqsubseteq Y\ i))$ 

```

```

lemma chain0 [simp]: chain Y  $\implies Y\ 0 = \text{false}$ 
  by (simp add: chain-def)

```



```

lemma chainI:
  assumes  $Y\ 0 = \text{false} \wedge i. Y\ (\text{Suc}\ i) \sqsubseteq Y\ i$ 
  shows chain  $Y$ 
  using assms by (auto simp add: chain-def)

lemma chainE:
  assumes chain  $Y \wedge i. \llbracket Y\ 0 = \text{false}; Y\ (\text{Suc}\ i) \sqsubseteq Y\ i \rrbracket \implies P$ 
  shows  $P$ 
  using assms by (simp add: chain-def)

lemma L274:
  assumes  $\forall n. ((E\ n \wedge_p X) = (E\ n \wedge Y))$ 
  shows  $(\bigcap (\text{range}\ E) \wedge X) = (\bigcap (\text{range}\ E) \wedge Y)$ 
  using assms by (pred-auto)

```

Constructive chains

```

definition constr ::
  ('a upred  $\Rightarrow$  'a upred)  $\Rightarrow$  'a chain  $\Rightarrow$  bool where
  constr  $F\ E \longleftrightarrow \text{chain}\ E \wedge (\forall X\ n. ((F(X) \wedge E(n+1)) = (F(X \wedge E(n)) \wedge E(n+1))))$ 

```

```

lemma constrI:
  assumes chain  $E \wedge X\ n. ((F(X) \wedge E(n+1)) = (F(X \wedge E(n)) \wedge E(n+1)))$ 
  shows constr  $F\ E$ 
  using assms by (auto simp add: constr-def)

```

This lemma gives a way of showing that there is a unique fixed-point when the predicate function can be built using a constructive function F over an approximation chain E

```

lemma chain-pred-terminates:
  assumes constr  $F\ E$  mono  $F$ 
  shows  $(\bigcap (\text{range}\ E) \wedge \mu\ F) = (\bigcap (\text{range}\ E) \wedge \nu\ F)$ 
proof -
  from assms have  $\forall n. (E\ n \wedge \mu\ F) = (E\ n \wedge \nu\ F)$ 
  proof (rule-tac allI)
    fix  $n$ 
    from assms show  $(E\ n \wedge \mu\ F) = (E\ n \wedge \nu\ F)$ 
    proof (induct  $n$ )
      case 0 thus ?case by (simp add: constr-def)
    next
      case (Suc  $n$ )
      note hyp = this
      thus ?case
      proof -
        have  $(E\ (n+1) \wedge \mu\ F) = (E\ (n+1) \wedge F\ (\mu\ F))$ 
        using gfp-unfold[OF hyp(3), THEN sym] by (simp add: constr-def)
        also from hyp have  $\dots = (E\ (n+1) \wedge F\ (E\ n \wedge \mu\ F))$ 
        by (metis conj-comm constr-def)
        also from hyp have  $\dots = (E\ (n+1) \wedge F\ (E\ n \wedge \nu\ F))$ 
        by simp
        also from hyp have  $\dots = (E\ (n+1) \wedge \nu\ F)$ 
        by (metis (no-types, lifting) conj-comm constr-def lfp-unfold)
        ultimately show ?thesis
        by simp
      qed
    qed
  qed

```

```

qed
thus ?thesis
  by (auto intro: L274)
qed

```

**theorem** *constr-fp-unig*:  
**assumes** *constr F E mono F*  $\sqcap$  (*range E*) = *C*  
**shows**  $(C \wedge \mu F) = (C \wedge \nu F)$   
**using** *assms(1) assms(2) assms(3) chain-pred-terminates* **by** *blast*

### 16.3 Noetherian Induction Instantiation

Contribution from Yakoub Nemouchi. The following generalization was used by Tobias Nipkow and Peter Lammich in *Refine\_Monadic*

**lemma** *wf-fixp-uinduct-pure-ueq-gen*:  
**assumes** *fixp-unfold*:  $fp\ B = B\ (fp\ B)$   
**and**  $WF$ : *wf R*  
**and** *induct-step*:  
 $\bigwedge f\ st.\ [\bigwedge st'. (st', st) \in R \implies (((Pre \wedge [e]_{<=u} \ll st' \gg) \Rightarrow Post) \sqsubseteq f)]$   
 $\implies fp\ B = f \implies ((Pre \wedge [e]_{<=u} \ll st \gg) \Rightarrow Post) \sqsubseteq (B\ f)$   
**shows**  $((Pre \Rightarrow Post) \sqsubseteq fp\ B)$   
**proof** –  
{ **fix** *st*  
**have**  $((Pre \wedge [e]_{<=u} \ll st \gg) \Rightarrow Post) \sqsubseteq (fp\ B)$   
**using** *WF* **proof** (*induction rule: wf-induct-rule*)  
**case** (*less x*)  
**hence**  $(Pre \wedge [e]_{<=u} \ll x \gg \Rightarrow Post) \sqsubseteq B\ (fp\ B)$   
**by** (*rule induct-step, rel-blast, simp*)  
**then show** ?*case*  
**using** *fixp-unfold* **by** *auto*  
**qed**  
}  
**thus** ?*thesis*  
**by** *pred-simp*  
**qed**

The next lemma shows that using substitution also work. However it is not that generic nor practical for proof automation ...

**lemma** *refine-ustbst-to-ueq*:  
 $vwb\text{-}lens\ E \implies (Pre \Rightarrow Post) \ll [st']_{>=E} \sqsubseteq f \ll [st']_{>=E} = (((Pre \wedge \$E =_u \ll st' \gg) \Rightarrow Post) \sqsubseteq f)$   
**by** (*rel-auto, metis vwb-lens-wb wb-lens.get-put*)

By instantiation of  $\ll [fp\ ?B = ?B\ (?fp\ ?B); wf\ ?R; \bigwedge f\ st.\ [\bigwedge st'. (st', st) \in ?R \implies (?Pre \wedge [?e]_{<=u} \ll st' \gg \Rightarrow ?Post) \sqsubseteq f; ?fp\ ?B = f] \implies (?Pre \wedge [?e]_{<=u} \ll st \gg \Rightarrow ?Post) \sqsubseteq ?B\ f] \implies (?Pre \Rightarrow ?Post) \sqsubseteq ?fp\ ?B$  with  $\mu$  and lifting of the well-founded relation we have ...

**lemma** *mu-rec-total-pure-rule*:  
**assumes**  $WF$ : *wf R*  
**and**  $M$ : *mono B*  
**and** *induct-step*:  
 $\bigwedge f\ st.\ [((Pre \wedge ([e]_{<=u} \ll st \gg)_u \in_u \ll R \gg \Rightarrow Post) \sqsubseteq f)]$   
 $\implies \mu\ B = f \implies (Pre \wedge [e]_{<=u} \ll st \gg \Rightarrow Post) \sqsubseteq (B\ f)$   
**shows**  $(Pre \Rightarrow Post) \sqsubseteq \mu\ B$   
**proof** (*rule wf-fixp-uinduct-pure-ueq-gen* [**where**  $fp=\mu$  **and**  $Pre=Pre$  **and**  $B=B$  **and**  $R=R$  **and**  $e=e$ ])  
**show**  $\mu\ B = B\ (\mu\ B)$

by (simp add: M def-gfp-unfold)  
 show wf R  
 by (fact WF)  
 show  $\bigwedge f st. (\bigwedge st'. (st', st) \in R \implies (Pre \wedge [e]_{<} =_u \ll st' \gg \Rightarrow Post) \sqsubseteq f) \implies$   
 $\mu B = f \implies$   
 $(Pre \wedge [e]_{<} =_u \ll st \gg \Rightarrow Post) \sqsubseteq B f$   
 by (rule induct-step, rel-simp, simp)  
 qed

**lemma** nu-rec-total-pure-rule:

assumes WF: wf R  
 and M: mono B  
 and induct-step:  
 $\bigwedge f st. \llbracket (Pre \wedge ([e]_{<}, \ll st \gg)_u \in_u \ll R \gg \Rightarrow Post) \sqsubseteq f \rrbracket$   
 $\implies \nu B = f \implies (Pre \wedge [e]_{<} =_u \ll st \gg \Rightarrow Post) \sqsubseteq (B f)$   
 shows  $(Pre \Rightarrow Post) \sqsubseteq \nu B$   
**proof** (rule wf-fixp-uinduct-pure-ueq-gen[where fp= $\nu$  and Pre=Pre and B=B and R=R and e=e])  
 show  $\nu B = B (\nu B)$   
 by (simp add: M def-lfp-unfold)  
 show wf R  
 by (fact WF)  
 show  $\bigwedge f st. (\bigwedge st'. (st', st) \in R \implies (Pre \wedge [e]_{<} =_u \ll st' \gg \Rightarrow Post) \sqsubseteq f) \implies$   
 $\nu B = f \implies$   
 $(Pre \wedge [e]_{<} =_u \ll st \gg \Rightarrow Post) \sqsubseteq B f$   
 by (rule induct-step, rel-simp, simp)  
 qed

Since  $B (Pre \wedge ([E]_{<}, \ll st \gg)_u \in_u \ll R \gg \Rightarrow Post) \sqsubseteq B (\mu B)$  and *mono B*, thus,  $\llbracket wf ?R; Monotonic ?B; \bigwedge f st. \llbracket (?Pre \wedge ([?e]_{<}, \ll st \gg)_u \in_u \ll ?R \gg \Rightarrow ?Post) \sqsubseteq f; \mu ?B = f \rrbracket \implies (?Pre \wedge [?e]_{<} =_u \ll st \gg \Rightarrow ?Post) \sqsubseteq ?B f \rrbracket \implies (?Pre \Rightarrow ?Post) \sqsubseteq \mu ?B$  can be expressed as follows

**lemma** mu-rec-total-utp-rule:

assumes WF: wf R  
 and M: mono B  
 and induct-step:  
 $\bigwedge st. (Pre \wedge [e]_{<} =_u \ll st \gg \Rightarrow Post) \sqsubseteq (B ((Pre \wedge ([e]_{<}, \ll st \gg)_u \in_u \ll R \gg \Rightarrow Post)))$   
 shows  $(Pre \Rightarrow Post) \sqsubseteq \mu B$   
**proof** (rule mu-rec-total-pure-rule[where R=R and e=e], simp-all add: assms)  
 show  $\bigwedge f st. (Pre \wedge ([e]_{<}, \ll st \gg)_u \in_u \ll R \gg \Rightarrow Post) \sqsubseteq f \implies \mu B = f \implies (Pre \wedge [e]_{<} =_u \ll st \gg \Rightarrow$   
 $Post) \sqsubseteq B f$   
 by (simp add: M induct-step monoD order-subst2)  
 qed

**lemma** nu-rec-total-utp-rule:

assumes WF: wf R  
 and M: mono B  
 and induct-step:  
 $\bigwedge st. (Pre \wedge [e]_{<} =_u \ll st \gg \Rightarrow Post) \sqsubseteq (B ((Pre \wedge ([e]_{<}, \ll st \gg)_u \in_u \ll R \gg \Rightarrow Post)))$   
 shows  $(Pre \Rightarrow Post) \sqsubseteq \nu B$   
**proof** (rule nu-rec-total-pure-rule[where R=R and e=e], simp-all add: assms)  
 show  $\bigwedge f st. (Pre \wedge ([e]_{<}, \ll st \gg)_u \in_u \ll R \gg \Rightarrow Post) \sqsubseteq f \implies \nu B = f \implies (Pre \wedge [e]_{<} =_u \ll st \gg \Rightarrow$   
 $Post) \sqsubseteq B f$   
 by (simp add: M induct-step monoD order-subst2)  
 qed

end

## 17 Sequent Calculus

**theory** *utp-sequent*

**imports** *utp-pred-laws*

**begin**

**definition** *sequent* :: ' $\alpha$  upred  $\Rightarrow$  ' $\alpha$  upred  $\Rightarrow$  bool (**infixr**  $\Vdash$  15) **where**  
 $[upred-defs]:$  *sequent*  $P \ Q = (Q \sqsubseteq P)$

**abbreviation** *sequent-triv* ( $\Vdash$  - [15] 15) **where**  $\Vdash P \equiv (true \Vdash P)$

**translations**

$\Vdash P \leq true \Vdash P$

**lemma** *sTrue*:  $P \Vdash true$

**by** *pred-auto*

**lemma** *sAx*:  $P \Vdash P$

**by** *pred-auto*

**lemma** *sNotI*:  $\Gamma \wedge P \Vdash false \implies \Gamma \Vdash \neg P$

**by** *pred-auto*

**lemma** *sConjI*:  $\llbracket \Gamma \Vdash P; \Gamma \Vdash Q \rrbracket \implies \Gamma \Vdash P \wedge Q$

**by** *pred-auto*

**lemma** *sImplI*:  $\llbracket (\Gamma \wedge P) \Vdash Q \rrbracket \implies \Gamma \Vdash (P \Rightarrow Q)$

**by** *pred-auto*

**end**

## 18 Relational Calculus Laws

**theory** *utp-rel-laws*

**imports**

*utp-rel*

*utp-recursion*

**begin**

### 18.1 Conditional Laws

**lemma** *comp-cond-left-distr*:

$((P \triangleleft b \triangleright_r Q) ;; R) = ((P ;; R) \triangleleft b \triangleright_r (Q ;; R))$

**by** (*rel-auto*)

**lemma** *cond-seq-left-distr*:

$out\alpha \# b \implies ((P \triangleleft b \triangleright Q) ;; R) = ((P ;; R) \triangleleft b \triangleright (Q ;; R))$

**by** (*rel-auto*)

**lemma** *cond-seq-right-distr*:

$in\alpha \# b \implies (P ;; (Q \triangleleft b \triangleright R)) = ((P ;; Q) \triangleleft b \triangleright (P ;; R))$

**by** (*rel-auto*)

Alternative expression of conditional using assumptions and choice

**lemma** *rcond-rassume-expand*:  $P \triangleleft b \triangleright_r Q = ([b]^\top ;; P) \sqcap ([(\neg b)]^\top ;; Q)$

by (rel-auto)

## 18.2 Precondition and Postcondition Laws

**theorem** *precond-equiv*:

$$P = (P ;; \text{true}) \longleftrightarrow (\text{out}\alpha \nmid P)$$

by (rel-auto)

**theorem** *postcond-equiv*:

$$P = (\text{true} ;; P) \longleftrightarrow (\text{in}\alpha \nmid P)$$

by (rel-auto)

**lemma** *precond-right-unit*:  $\text{out}\alpha \nmid p \implies (p ;; \text{true}) = p$

by (metis *precond-equiv*)

**lemma** *postcond-left-unit*:  $\text{in}\alpha \nmid p \implies (\text{true} ;; p) = p$

by (metis *postcond-equiv*)

**theorem** *precond-left-zero*:

assumes  $\text{out}\alpha \nmid p \implies p \neq \text{false}$

shows  $(\text{true} ;; p) = \text{true}$

using *assms* by (rel-auto)

**theorem** *feasible-iff-true-right-zero*:

$$P ;; \text{true} = \text{true} \longleftrightarrow \exists \text{out}\alpha \cdot P$$

by (rel-auto)

## 18.3 Sequential Composition Laws

**lemma** *segr-assoc*:  $(P ;; Q) ;; R = P ;; (Q ;; R)$

by (rel-auto)

**lemma** *segr-left-unit* [*simp*]:

$$II ;; P = P$$

by (rel-auto)

**lemma** *segr-right-unit* [*simp*]:

$$P ;; II = P$$

by (rel-auto)

**lemma** *segr-left-zero* [*simp*]:

$$\text{false} ;; P = \text{false}$$

by *pred-auto*

**lemma** *segr-right-zero* [*simp*]:

$$P ;; \text{false} = \text{false}$$

by *pred-auto*

**lemma** *impl-segr-mono*:  $\llbracket P \Rightarrow Q; R \Rightarrow S \rrbracket \implies (P ;; R) \Rightarrow (Q ;; S)$

by (*pred-blast*)

**lemma** *segr-mono*:

$$\llbracket P_1 \sqsubseteq P_2; Q_1 \sqsubseteq Q_2 \rrbracket \implies (P_1 ;; Q_1) \sqsubseteq (P_2 ;; Q_2)$$

by (*rel-blast*)

**lemma** *segr-monotonic*:

$\llbracket \text{mono } P; \text{mono } Q \rrbracket \implies \text{mono } (\lambda X. P X ;; Q X)$   
**by** (*simp add: mono-def, rel-blast*)

**lemma** *Monotonic-seqr-tail* [*closure*]:  
**assumes** *Monotonic F*  
**shows** *Monotonic*  $(\lambda X. P ;; F(X))$   
**by** (*simp add: assms monoD monoI seqr-mono*)

**lemma** *seqr-exists-left*:  
 $((\exists \$x \cdot P) ;; Q) = (\exists \$x \cdot (P ;; Q))$   
**by** (*rel-auto*)

**lemma** *seqr-exists-right*:  
 $(P ;; (\exists \$x' \cdot Q)) = (\exists \$x' \cdot (P ;; Q))$   
**by** (*rel-auto*)

**lemma** *seqr-or-distl*:  
 $((P \vee Q) ;; R) = ((P ;; R) \vee (Q ;; R))$   
**by** (*rel-auto*)

**lemma** *seqr-or-distr*:  
 $(P ;; (Q \vee R)) = ((P ;; Q) \vee (P ;; R))$   
**by** (*rel-auto*)

**lemma** *seqr-inf-distl*:  
 $((P \sqcap Q) ;; R) = ((P ;; R) \sqcap (Q ;; R))$   
**by** (*rel-auto*)

**lemma** *seqr-inf-distr*:  
 $(P ;; (Q \sqcap R)) = ((P ;; Q) \sqcap (P ;; R))$   
**by** (*rel-auto*)

**lemma** *seqr-and-distr-ufunc*:  
*ufunctional P*  $\implies (P ;; (Q \wedge R)) = ((P ;; Q) \wedge (P ;; R))$   
**by** (*rel-auto*)

**lemma** *seqr-and-distl-ujnj*:  
*ujnj R*  $\implies ((P \wedge Q) ;; R) = ((P ;; R) \wedge (Q ;; R))$   
**by** (*rel-auto*)

**lemma** *seqr-unfold*:  
 $(P ;; Q) = (\exists v \cdot P \llbracket \llbracket v \rrbracket / \$v' \rrbracket \wedge Q \llbracket \llbracket v \rrbracket / \$v \rrbracket)$   
**by** (*rel-auto*)

**lemma** *seqr-middle*:  
**assumes** *vwb-lens x*  
**shows**  $(P ;; Q) = (\exists v \cdot P \llbracket \llbracket v \rrbracket / \$x' \rrbracket ;; Q \llbracket \llbracket v \rrbracket / \$x \rrbracket)$   
**using** *assms*  
**by** (*rel-auto', metis vwb-lens-wb wb-lens.source-stability*)

**lemma** *seqr-left-one-point*:  
**assumes** *vwb-lens x*  
**shows**  $((P \wedge \$x' =_u \llbracket v \rrbracket) ;; Q) = (P \llbracket \llbracket v \rrbracket / \$x' \rrbracket ;; Q \llbracket \llbracket v \rrbracket / \$x \rrbracket)$   
**using** *assms*  
**by** (*rel-auto, metis vwb-lens-wb wb-lens.get-put*)

**lemma** *seqr-right-one-point*:

**assumes** *vwb-lens*  $x$   
**shows**  $(P ;; (\$x =_u \ll v \gg \wedge Q)) = (P[\ll v \gg / \$x'] ;; Q[\ll v \gg / \$x])$   
**using** *assms*  
**by** (*rel-auto*, *metis vwb-lens-wb wb-lens.get-put*)

**lemma** *seqr-left-one-point-true*:

**assumes** *vwb-lens*  $x$   
**shows**  $((P \wedge \$x') ;; Q) = (P[\text{true} / \$x'] ;; Q[\text{true} / \$x])$   
**by** (*metis assms seqr-left-one-point true-alt-def upred-eq-true*)

**lemma** *seqr-left-one-point-false*:

**assumes** *vwb-lens*  $x$   
**shows**  $((P \wedge \neg \$x') ;; Q) = (P[\text{false} / \$x'] ;; Q[\text{false} / \$x])$   
**by** (*metis assms false-alt-def seqr-left-one-point upred-eq-false*)

**lemma** *seqr-right-one-point-true*:

**assumes** *vwb-lens*  $x$   
**shows**  $(P ;; (\$x \wedge Q)) = (P[\text{true} / \$x'] ;; Q[\text{true} / \$x])$   
**by** (*metis assms seqr-right-one-point true-alt-def upred-eq-true*)

**lemma** *seqr-right-one-point-false*:

**assumes** *vwb-lens*  $x$   
**shows**  $(P ;; (\neg \$x \wedge Q)) = (P[\text{false} / \$x'] ;; Q[\text{false} / \$x])$   
**by** (*metis assms false-alt-def seqr-right-one-point upred-eq-false*)

**lemma** *seqr-insert-ident-left*:

**assumes** *vwb-lens*  $x$   $\$x' \# P$   $\$x \# Q$   
**shows**  $((\$x' =_u \$x \wedge P) ;; Q) = (P ;; Q)$   
**using** *assms*  
**by** (*rel-simp*, *meson vwb-lens-wb wb-lens-weak weak-lens.put-get*)

**lemma** *seqr-insert-ident-right*:

**assumes** *vwb-lens*  $x$   $\$x' \# P$   $\$x \# Q$   
**shows**  $(P ;; (\$x' =_u \$x \wedge Q)) = (P ;; Q)$   
**using** *assms*  
**by** (*rel-simp*, *metis (no-types, hide-lams) vwb-lens-def wb-lens-def weak-lens.put-get*)

**lemma** *seq-var-ident-lift*:

**assumes** *vwb-lens*  $x$   $\$x' \# P$   $\$x \# Q$   
**shows**  $((\$x' =_u \$x \wedge P) ;; (\$x' =_u \$x \wedge Q)) = (\$x' =_u \$x \wedge (P ;; Q))$   
**using** *assms* **by** (*rel-auto'*, *metis (no-types, lifting) vwb-lens-wb wb-lens-weak weak-lens.put-get*)

**lemma** *seqr-bool-split*:

**assumes** *vwb-lens*  $x$   
**shows**  $P ;; Q = (P[\text{true} / \$x'] ;; Q[\text{true} / \$x] \vee P[\text{false} / \$x'] ;; Q[\text{false} / \$x])$   
**using** *assms*  
**by** (*subst seqr-middle[of x]*, *simp-all*)

**lemma** *cond-inter-var-split*:

**assumes** *vwb-lens*  $x$   
**shows**  $(P \triangleleft \$x' \triangleright Q) ;; R = (P[\text{true} / \$x'] ;; R[\text{true} / \$x] \vee Q[\text{false} / \$x'] ;; R[\text{false} / \$x])$

**proof** –

**have**  $(P \triangleleft \$x' \triangleright Q) ;; R = ((\$x' \wedge P) ;; R \vee (\neg \$x' \wedge Q) ;; R)$

by (simp add: cond-def seqr-or-distl)  
 also have ... =  $((P \wedge \$x') ;; R \vee (Q \wedge \neg \$x') ;; R)$   
 by (rel-auto)  
 also have ... =  $(P \llbracket \text{true}/\$x' \rrbracket ;; R \llbracket \text{true}/\$x \rrbracket \vee Q \llbracket \text{false}/\$x' \rrbracket ;; R \llbracket \text{false}/\$x \rrbracket)$   
 by (simp add: seqr-left-one-point-true seqr-left-one-point-false assms)  
 finally show ?thesis .  
 qed

**theorem seqr-pre-transfer:**  $\text{in}\alpha \# q \implies ((P \wedge q) ;; R) = (P ;; (q^- \wedge R))$   
 by (rel-auto)

**theorem seqr-pre-transfer':**  
 $((P \wedge \lceil q \rceil_{>}) ;; R) = (P ;; (\lceil q \rceil_{<} \wedge R))$   
 by (rel-auto)

**theorem seqr-post-out:**  $\text{in}\alpha \# r \implies (P ;; (Q \wedge r)) = ((P ;; Q) \wedge r)$   
 by (rel-blast)

**lemma seqr-post-var-out:**  
 fixes  $x :: (\text{bool} \implies 'a)$   
 shows  $(P ;; (Q \wedge \$x')) = ((P ;; Q) \wedge \$x')$   
 by (rel-auto)

**theorem seqr-post-transfer:**  $\text{out}\alpha \# q \implies (P ;; (q \wedge R)) = ((P \wedge q^-) ;; R)$   
 by (rel-auto)

**lemma seqr-pre-out:**  $\text{out}\alpha \# p \implies ((p \wedge Q) ;; R) = (p \wedge (Q ;; R))$   
 by (rel-blast)

**lemma seqr-pre-var-out:**  
 fixes  $x :: (\text{bool} \implies 'a)$   
 shows  $((\$x \wedge P) ;; Q) = (\$x \wedge (P ;; Q))$   
 by (rel-auto)

**lemma seqr-true-lemma:**  
 $(P = (\neg ((\neg P) ;; \text{true}))) = (P = (P ;; \text{true}))$   
 by (rel-auto)

**lemma seqr-to-conj:**  $\llbracket \text{out}\alpha \# P; \text{in}\alpha \# Q \rrbracket \implies (P ;; Q) = (P \wedge Q)$   
 by (metis postcond-left-unit seqr-pre-out utp-pred-laws.inf-top.right-neutral)

**lemma shEx-lift-seq-1 [uquant-lift]:**  
 $((\exists x \cdot P x) ;; Q) = (\exists x \cdot (P x ;; Q))$   
 by rel-auto

**lemma shEx-mem-lift-seq-1 [uquant-lift]:**  
 assumes  $\text{out}\alpha \# A$   
 shows  $((\exists x \in A \cdot P x) ;; Q) = (\exists x \in A \cdot (P x ;; Q))$   
 using assms by rel-blast

**lemma shEx-lift-seq-2 [uquant-lift]:**  
 $(P ;; (\exists x \cdot Q x)) = (\exists x \cdot (P ;; Q x))$   
 by rel-auto

**lemma shEx-mem-lift-seq-2 [uquant-lift]:**



**assumes**  $in\alpha \# A$   
**shows**  $(P ;; (\exists x \in A \cdot Q\ x)) = (\exists x \in A \cdot (P ;; Q\ x))$   
**using** *assms* **by** *rel-blast*

## 18.4 Iterated Sequential Composition Laws

**lemma** *iter-seqr-nil* [*simp*]:  $(;; i : [] \cdot P(i)) = II$   
**by** (*simp add: seqr-iter-def*)

**lemma** *iter-seqr-cons* [*simp*]:  $(;; i : (x \# xs) \cdot P(i)) = P(x) ;; (;; i : xs \cdot P(i))$   
**by** (*simp add: seqr-iter-def*)

## 18.5 Quantale Laws

**lemma** *seq-Sup-distl*:  $P ;; (\bigcap A) = (\bigcap_{Q \in A} P ;; Q)$   
**by** (*transfer, auto*)

**lemma** *seq-Sup-distr*:  $(\bigcap A) ;; Q = (\bigcap_{P \in A} P ;; Q)$   
**by** (*transfer, auto*)

**lemma** *seq-UNIF-distl*:  $P ;; (\bigcap_{Q \in A} F(Q)) = (\bigcap_{Q \in A} P ;; F(Q))$   
**by** (*simp add: UNIF-as-Sup-collect seq-Sup-distl*)

**lemma** *seq-UNIF-distl'*:  $P ;; (\bigcap Q \cdot F(Q)) = (\bigcap Q \cdot P ;; F(Q))$   
**by** (*metis UNIF-mem-UNIV seq-UNIF-distl*)

**lemma** *seq-UNIF-distr*:  $(\bigcap_{P \in A} F(P)) ;; Q = (\bigcap_{P \in A} P \cdot F(P) ;; Q)$   
**by** (*simp add: UNIF-as-Sup-collect seq-Sup-distr*)

**lemma** *seq-UNIF-distr'*:  $(\bigcap P \cdot F(P)) ;; Q = (\bigcap P \cdot F(P) ;; Q)$   
**by** (*metis UNIF-mem-UNIV seq-UNIF-distr*)

**lemma** *seq-SUP-distl*:  $P ;; (\bigcap_{i \in A} Q(i)) = (\bigcap_{i \in A} P ;; Q(i))$   
**by** (*metis image-image seq-Sup-distl*)

**lemma** *seq-SUP-distr*:  $(\bigcap_{i \in A} P(i)) ;; Q = (\bigcap_{i \in A} P(i) ;; Q)$   
**by** (*simp add: seq-Sup-distr*)

## 18.6 Skip Laws

**lemma** *cond-skip*:  $out\alpha \# b \implies (b \wedge II) = (II \wedge b^-)$   
**by** (*rel-auto*)

**lemma** *pre-skip-post*:  $([b]_< \wedge II) = (II \wedge [b]_>)$   
**by** (*rel-auto*)

**lemma** *skip-var*:  
**fixes**  $x :: (bool \implies 'a)$   
**shows**  $(\$x \wedge II) = (II \wedge \$x')$   
**by** (*rel-auto*)

**lemma** *skip-r-unfold*:  
 $vwb\text{-}lens\ x \implies II = (\$x' =_u \$x \wedge II \upharpoonright_\alpha x)$   
**by** (*rel-simp, metis mwb-lens.put-put vwb-lens-mwb vwb-lens-wb wb-lens.get-put*)

**lemma** *skip-r-alpha-eq*:

$II = (\$v' =_u \$v)$   
**by** (*rel-auto*)

**lemma** *skip-ra-unfold*:  
 $II_{x;y} = (\$x' =_u \$x \wedge II_y)$   
**by** (*rel-auto*)

**lemma** *skip-res-as-ra*:  
 $\llbracket vwb\text{-}lens\ y; x +_L y \approx_L 1_L; x \bowtie y \rrbracket \implies II|_{\alpha} x = II_y$   
**apply** (*rel-auto*)  
**apply** (*metis* (*no-types*, *lifting*) *lens-indep-def*)  
**apply** (*metis* *vwb-lens.put-eq*)  
**done**

## 18.7 Assignment Laws

**lemma** *assigns-subst* [*usubst*]:  
 $[\sigma]_s \dagger \langle \varrho \rangle_a = \langle \varrho \circ \sigma \rangle_a$   
**by** (*rel-auto*)

**lemma** *assigns-r-comp*:  $(\langle \sigma \rangle_a ;; P) = ([\sigma]_s \dagger P)$   
**by** (*rel-auto*)

**lemma** *assigns-r-feasible*:  
 $(\langle \sigma \rangle_a ;; true) = true$   
**by** (*rel-auto*)

**lemma** *assign-subst* [*usubst*]:  
 $\llbracket mwb\text{-}lens\ x; mwb\text{-}lens\ y \rrbracket \implies [\$x \mapsto_s [u]_{<}] \dagger (y := v) = (x, y) := (u, [x \mapsto_s u] \dagger v)$   
**by** (*rel-auto*)

**lemma** *assign-vacuous-skip*:  
**assumes** *vwb-lens* *x*  
**shows**  $(x := \&x) = II$   
**using** *assms* **by** *rel-auto*

The following law shows the case for the above law when *x* is only mainly-well behaved. We require that the state is one of those in which *x* is well defined using and assumption.

**lemma** *assign-vacuous-assume*:  
**assumes** *mwb-lens* *x*  
**shows**  $[(\&\mathbf{v} \in_u \llbracket \mathcal{S}_x \rrbracket)]^\top ;; (x := \&x) = [(\&\mathbf{v} \in_u \llbracket \mathcal{S}_x \rrbracket)]^\top$   
**using** *assms* **by** *rel-auto*

**lemma** *assign-simultaneous*:  
**assumes** *vwb-lens* *y*  $x \bowtie y$   
**shows**  $(x, y) := (e, \&y) = (x := e)$   
**by** (*simp* *add: assms usubst-upd-comm usubst-upd-var-id*)

**lemma** *assigns-idem*: *mwb-lens* *x*  $\implies (x, x) := (u, v) = (x := v)$   
**by** (*simp* *add: usubst*)

**lemma** *assigns-comp*:  $(\langle f \rangle_a ;; \langle g \rangle_a) = \langle g \circ f \rangle_a$   
**by** (*simp* *add: assigns-r-comp usubst*)

**lemma** *assigns-cond*:  $(\langle f \rangle_a \triangleleft b \triangleright_r \langle g \rangle_a) = \langle f \triangleleft b \triangleright_s g \rangle_a$

by (rel-auto)

**lemma** *assigns-r-conv*:

$\text{bij } f \implies \langle f \rangle_a^- = \langle \text{inv } f \rangle_a$

by (rel-auto, simp-all add: bij-is-inj bij-is-surj surj-f-inv-f)

**lemma** *assign-pred-transfer*:

fixes  $x :: ('a \implies 'a)$

assumes  $\$x \# b \text{ out } \alpha \# b$

shows  $(b \wedge x := v) = (x := v \wedge b^-)$

using assms by (rel-blast)

**lemma** *assign-r-comp*:  $x := u ;; P = P[[u]_{<}/\$x]$

by (simp add: assigns-r-comp usubst alpha)

**lemma** *assign-test*:  $\text{mwb-lens } x \implies (x := \llbracket u \rrbracket ;; x := \llbracket v \rrbracket) = (x := \llbracket v \rrbracket)$

by (simp add: assigns-comp usubst)

**lemma** *assign-twice*:  $\llbracket \text{mwb-lens } x; x \# f \rrbracket \implies (x := e ;; x := f) = (x := f)$

by (simp add: assigns-comp usubst unrest)

**lemma** *assign-commute*:

assumes  $x \bowtie y \text{ } x \# f \text{ } y \# e$

shows  $(x := e ;; y := f) = (y := f ;; x := e)$

using assms

by (rel-simp, simp-all add: lens-indep-comm)

**lemma** *assign-cond*:

fixes  $x :: ('a \implies 'a)$

assumes  $\text{out } \alpha \# b$

shows  $(x := e ;; (P \triangleleft b \triangleright Q)) = ((x := e ;; P) \triangleleft (b[[e]_{<}/\$x]) \triangleright (x := e ;; Q))$

by (rel-auto)

**lemma** *assign-rcond*:

fixes  $x :: ('a \implies 'a)$

assumes  $\text{out } \alpha \# b$

shows  $(x := e ;; (P \triangleleft b \triangleright_r Q)) = ((x := e ;; P) \triangleleft (b[[e/x]]) \triangleright_r (x := e ;; Q))$

by (rel-auto)

**lemma** *assign-r-alt-def*:

fixes  $x :: ('a \implies 'a)$

shows  $x := v = II[[v]_{<}/\$x]$

by (rel-auto)

**lemma** *assigns-r-ufunc*: *ufunctional*  $\langle f \rangle_a$

by (rel-auto)

**lemma** *assigns-r-uinj*:  $\text{inj } f \implies \text{uinj } \langle f \rangle_a$

by (rel-simp, simp add: inj-eq)

**lemma** *assigns-r-swap-uinj*:

$\llbracket \text{vwb-lens } x; \text{vwb-lens } y; x \bowtie y \rrbracket \implies \text{uinj } ((x, y) := (\&y, \&x))$

by (metis assigns-r-uinj pr-var-def swap-usubst-inj)

**lemma** *assign-unfold*:

$\text{vwb-lens } x \implies (x := v) = (\$x' =_u [v]_{<} \wedge II \downarrow_\alpha x)$

**apply** (*rel-auto*, *auto simp add: comp-def*)  
**using** *vwb-lens.put-eq* **by** *fastforce*

## 18.8 Non-deterministic Assignment Laws

**lemma** *nd-assign-comp*:  
 $x \bowtie y \implies x := * ;; y := * = x, y := *$   
**apply** (*rel-auto*) **using** *lens-indep-comm* **by** *fastforce+*

**lemma** *nd-assign-assign*:  
 $\llbracket \text{vwb-lens } x; x \# e \rrbracket \implies x := * ;; x := e = x := e$   
**by** (*rel-auto*)

## 18.9 Converse Laws

**lemma** *convr-invol* [*simp*]:  $p^{- -} = p$   
**by** *pred-auto*

**lemma** *lit-convr* [*simp*]:  $\llbracket v \rrbracket^{-} = \llbracket v \rrbracket$   
**by** *pred-auto*

**lemma** *uivar-convr* [*simp*]:  
**fixes**  $x :: ('a \implies 'a)$   
**shows**  $(\$x)^{-} = \$x'$   
**by** *pred-auto*

**lemma** *uovar-convr* [*simp*]:  
**fixes**  $x :: ('a \implies 'a)$   
**shows**  $(\$x')^{-} = \$x$   
**by** *pred-auto*

**lemma** *uop-convr* [*simp*]:  $(uop\ f\ u)^{-} = uop\ f\ (u^{-})$   
**by** (*pred-auto*)

**lemma** *bop-convr* [*simp*]:  $(bop\ f\ u\ v)^{-} = bop\ f\ (u^{-})\ (v^{-})$   
**by** (*pred-auto*)

**lemma** *eq-convr* [*simp*]:  $(p =_u q)^{-} = (p^{-} =_u q^{-})$   
**by** (*pred-auto*)

**lemma** *not-convr* [*simp*]:  $(\neg p)^{-} = (\neg p^{-})$   
**by** (*pred-auto*)

**lemma** *disj-convr* [*simp*]:  $(p \vee q)^{-} = (q^{-} \vee p^{-})$   
**by** (*pred-auto*)

**lemma** *conj-convr* [*simp*]:  $(p \wedge q)^{-} = (q^{-} \wedge p^{-})$   
**by** (*pred-auto*)

**lemma** *seqr-convr* [*simp*]:  $(p ;; q)^{-} = (q^{-} ;; p^{-})$   
**by** (*rel-auto*)

**lemma** *pre-convr* [*simp*]:  $[p]_{<}^{-} = [p]_{>}$   
**by** (*rel-auto*)

**lemma** *post-convr* [*simp*]:  $[p]_{>}^{-} = [p]_{<}$

by (rel-auto)

## 18.10 Assertion and Assumption Laws

declare *sublens-def* [*lens-defs del*]

lemma *assume-false*:  $[false]^\top = false$   
by (rel-auto)

lemma *assume-true*:  $[true]^\top = II$   
by (rel-auto)

lemma *assume-seq*:  $[b]^\top ;; [c]^\top = [(b \wedge c)]^\top$   
by (rel-auto)

lemma *assert-false*:  $\{false\}_\perp = true$   
by (rel-auto)

lemma *assert-true*:  $\{true\}_\perp = II$   
by (rel-auto)

lemma *assert-seq*:  $\{b\}_\perp ;; \{c\}_\perp = \{(b \wedge c)\}_\perp$   
by (rel-auto)

## 18.11 Frame and Antiframe Laws

named-theorems *frame*

lemma *frame-all* [*frame*]:  $\Sigma:[P] = P$   
by (rel-auto)

lemma *frame-none* [*frame*]:  
 $\emptyset:[P] = (P \wedge II)$   
by (rel-auto)

lemma *frame-commute*:  
assumes  $\$y \# P \$y' \# P \$x \# Q \$x' \# Q x \bowtie y$   
shows  $x:[P] ;; y:[Q] = y:[Q] ;; x:[P]$   
apply (insert *assms*)  
apply (rel-auto)  
apply (rename-tac *s s' s<sub>0</sub>*)  
apply (subgoal-tac (*s*  $\oplus_L$  *s'* on *y*)  $\oplus_L$  *s<sub>0</sub>* on *x* = *s<sub>0</sub>*  $\oplus_L$  *s'* on *y*)  
  apply (metis *lens-indep-get lens-indep-sym lens-override-def*)  
  apply (simp add: *lens-indep.lens-put-comm lens-override-def*)  
  apply (rename-tac *s s' s<sub>0</sub>*)  
  apply (subgoal-tac *put<sub>y</sub>* (*put<sub>x</sub>* *s* (*get<sub>x</sub>* (*put<sub>x</sub>* *s<sub>0</sub>* (*get<sub>x</sub>* *s'*)))) (*get<sub>y</sub>* (*put<sub>y</sub>* *s* (*get<sub>y</sub>* *s<sub>0</sub>*)))  
    = *put<sub>x</sub>* *s<sub>0</sub>* (*get<sub>x</sub>* *s'*))  
  apply (metis *lens-indep-get lens-indep-sym*)  
  apply (metis *lens-indep.lens-put-comm*)  
done

lemma *frame-contract-RID*:  
assumes *vwb-lens* *x P* is *RID*(*x*) *x*  $\bowtie$  *y*  
shows  $(x;y):[P] = y:[P]$   
proof –  
  from *assms*(1,3) have  $(x;y):[RID(x)(P)] = y:[RID(x)(P)]$

```

    apply (rel-auto)
    apply (simp add: lens-indep.lens-put-comm)
    apply (metis (no-types) vwb-lens-wb wb-lens.get-put)
  done
thus ?thesis
  by (simp add: Healthy-if assms)
qed

lemma frame-miracle [simp]:
  x:[false] = false
  by (rel-auto)

lemma frame-skip [simp]:
  vwb-lens x  $\implies$  x:[II] = II
  by (rel-auto)

lemma frame-assign-in [frame]:
   $\llbracket \text{vwb-lens } a; x \subseteq_L a \rrbracket \implies a:[x := v] = x := v$ 
  by (rel-auto, simp-all add: lens-get-put-quasi-commute lens-put-of-quotient)

lemma frame-conj-true [frame]:
   $\llbracket \{ \$x, \$x' \} \Vdash P; \text{vwb-lens } x \rrbracket \implies (P \wedge x:[\text{true}]) = x:[P]$ 
  by (rel-auto)

lemma frame-is-assign [frame]:
  vwb-lens x  $\implies$  x:[ $\$x' =_u [v]_{<}$ ] = x := v
  by (rel-auto)

lemma frame-seq [frame]:
   $\llbracket \text{vwb-lens } x; \{ \$x, \$x' \} \Vdash P; \{ \$x, \$x' \} \Vdash Q \rrbracket \implies x:[P ;; Q] = x:[P] ;; x:[Q]$ 
  apply (rel-auto)
  apply (metis mwb-lens.put-put vwb-lens-mwb vwb-lens-wb wb-lens-def weak-lens.put-get)
  apply (metis mwb-lens.put-put vwb-lens-mwb)
  done

lemma frame-to-antiframe [frame]:
   $\llbracket x \bowtie y; x +_L y = 1_L \rrbracket \implies x:[P] = y:[P]$ 
  by (rel-auto, metis lens-indep-def, metis lens-indep-def surj-pair)

lemma rel-frex-miracle [frame]:
  a:[false]+ = false
  by (rel-auto)

lemma rel-frex-skip [frame]:
  vwb-lens a  $\implies$  a:[II]+ = II
  by (rel-auto)

lemma rel-frex-seq [frame]:
  vwb-lens a  $\implies$  a:[P ;; Q]+ = (a:[P]+ ;; a:[Q]+)
  apply (rel-auto)
  apply (rename-tac s s' s0)
  apply (rule-tac x=puta s s0 in exI)
  apply (auto)
  apply (metis mwb-lens.put-put vwb-lens-mwb)
  done

```

**lemma** *rel-frex-assigms* [frame]:  
 $vwb\text{-}lens\ a \implies a:[\langle\sigma\rangle_a]^+ = \langle\sigma \oplus_s a\rangle_a$   
**by** (*rel-auto*)

**lemma** *rel-frex-rcond* [frame]:  
 $a:[P \triangleleft b \triangleright_r Q]^+ = (a:[P]^+ \triangleleft b \oplus_p a \triangleright_r a:[Q]^+)$   
**by** (*rel-auto*)

**lemma** *rel-frex-commute*:  
 $x \bowtie y \implies x:[P]^+ ;; y:[Q]^+ = y:[Q]^+ ;; x:[P]^+$   
**apply** (*rel-auto*)  
**apply** (*rename-tac a c b*)  
**apply** (*subgoal-tac*  $\bigwedge b\ a.\ get_y\ (put_x\ b\ a) = get_y\ b$ )  
**apply** (*metis* (*no-types*, *hide-lams*) *lens-indep-comm lens-indep-get*)  
**apply** (*simp add: lens-indep.lens-put-irr2*)  
**apply** (*subgoal-tac*  $\bigwedge b\ c.\ get_x\ (put_y\ b\ c) = get_x\ b$ )  
**apply** (*subgoal-tac*  $\bigwedge b\ a.\ get_y\ (put_x\ b\ a) = get_y\ b$ )  
**apply** (*metis* (*mono-tags*, *lifting*) *lens-indep-comm*)  
**apply** (*simp-all add: lens-indep.lens-put-irr2*)  
**done**

**lemma** *antiframe-disj* [frame]:  $(x:[P] \vee x:[Q]) = x:[P \vee Q]$   
**by** (*rel-auto*)

**lemma** *antiframe-seq* [frame]:  
 $\llbracket vwb\text{-}lens\ x; \$x' \# P; \$x \# Q \rrbracket \implies (x:[P] ;; x:[Q]) = x:[P ;; Q]$   
**apply** (*rel-auto*)  
**apply** (*metis vwb-lens-wb wb-lens-def weak-lens.put-get*)  
**apply** (*metis vwb-lens-wb wb-lens.put-twice wb-lens-def weak-lens.put-get*)  
**done**

**lemma** *nameset-skip*:  $vwb\text{-}lens\ x \implies (ns\ x \cdot II) = II_x$   
**by** (*rel-auto*, *meson vwb-lens-wb wb-lens.get-put*)

**lemma** *nameset-skip-ra*:  $vwb\text{-}lens\ x \implies (ns\ x \cdot II_x) = II_x$   
**by** (*rel-auto*)

**declare** *sublens-def* [*lens-defs*]

## 18.12 While Loop Laws

**theorem** *while-unfold*:  
 $while\ b\ do\ P\ od = ((P ;; while\ b\ do\ P\ od) \triangleleft b \triangleright_r II)$   
**proof** –  
**have**  $m:mono\ (\lambda X.\ (P ;; X) \triangleleft b \triangleright_r II)$   
**by** (*auto intro: monoI segr-mono cond-mono*)  
**have**  $(while\ b\ do\ P\ od) = (\nu\ X \cdot (P ;; X) \triangleleft b \triangleright_r II)$   
**by** (*simp add: while-top-def*)  
**also have**  $\dots = ((P ;; (\nu\ X \cdot (P ;; X) \triangleleft b \triangleright_r II)) \triangleleft b \triangleright_r II)$   
**by** (*subst lfp-unfold, simp-all add: m*)  
**also have**  $\dots = ((P ;; while\ b\ do\ P\ od) \triangleleft b \triangleright_r II)$   
**by** (*simp add: while-top-def*)  
**finally show** *?thesis* .  
**qed**

**theorem** *while-false*:  $\text{while false do } P \text{ od} = II$   
 by (subst while-unfold, rel-auto)

**theorem** *while-true*:  $\text{while true do } P \text{ od} = \text{false}$   
 apply (simp add: while-top-def alpha)  
 apply (rule antisym)  
 apply (simp-all)  
 apply (rule lfp-lowerbound)  
 apply (rel-auto)  
 done

**theorem** *while-bot-unfold*:  
 $\text{while}_\perp b \text{ do } P \text{ od} = ((P ;; \text{while}_\perp b \text{ do } P \text{ od}) \triangleleft b \triangleright_r II)$

**proof** –

have  $m:\text{mono } (\lambda X. (P ;; X) \triangleleft b \triangleright_r II)$   
 by (auto intro: monoI segr-mono cond-mono)  
 have  $(\text{while}_\perp b \text{ do } P \text{ od}) = (\mu X \cdot (P ;; X) \triangleleft b \triangleright_r II)$   
 by (simp add: while-bot-def)  
 also have  $\dots = ((P ;; (\mu X \cdot (P ;; X) \triangleleft b \triangleright_r II)) \triangleleft b \triangleright_r II)$   
 by (subst gfp-unfold, simp-all add: m)  
 also have  $\dots = ((P ;; \text{while}_\perp b \text{ do } P \text{ od}) \triangleleft b \triangleright_r II)$   
 by (simp add: while-bot-def)  
 finally show ?thesis .

qed

**theorem** *while-bot-false*:  $\text{while}_\perp \text{false do } P \text{ od} = II$   
 by (simp add: while-bot-def mu-const alpha)

**theorem** *while-bot-true*:  $\text{while}_\perp \text{true do } P \text{ od} = (\mu X \cdot P ;; X)$   
 by (simp add: while-bot-def alpha)

An infinite loop with a feasible body corresponds to a program error (non-termination).

**theorem** *while-infinite*:  $P ;; \text{true}_h = \text{true} \implies \text{while}_\perp \text{true do } P \text{ od} = \text{true}$   
 apply (simp add: while-bot-true)  
 apply (rule antisym)  
 apply (simp)  
 apply (rule gfp-upperbound)  
 apply (simp)  
 done

## 18.13 Algebraic Properties

**interpretation** *upred-semiring*: *semiring-1*

where *times* = *segr* and *one* = *skip-r* and *zero* = *false<sub>h</sub>* and *plus* = *Lattices.sup*  
 by (unfold-locales, (rel-auto)+)

**declare** *upred-semiring.power-Suc* [simp del]

We introduce the power syntax derived from semirings

**abbreviation** *upower* :: ' $\alpha \text{ hrel} \Rightarrow \text{nat} \Rightarrow ' \alpha \text{ hrel}$  (infixr ^ 80) **where**  
 $\text{upower } P \ n \equiv \text{upred-semiring.power } P \ n$

**translations**

$P \wedge i \leq \text{CONST power.power } II \text{ op} ;; P \ i$   
 $P \wedge i \leq (\text{CONST power.power } II \text{ op} ;; P) \ i$



Set up transfer tactic for powers

```

lemma upower-rep-eq:
   $\llbracket P \wedge i \rrbracket_e = (\lambda b. b \in (\{p. \llbracket P \rrbracket_e p\} \wedge i))$ 
proof (induct i arbitrary: P)
  case 0
  then show ?case
    by (auto, rel-auto)
next
  case (Suc i)
  show ?case
    by (simp add: Suc segr.rep-eq relpow-commute upred-semiring.power-Suc)
qed

```

```

lemma upower-rep-eq-alt:
   $\llbracket power.power \langle id \rangle_a (;;) P i \rrbracket_e = (\lambda b. b \in (\{p. \llbracket P \rrbracket_e p\} \wedge i))$ 
  by (metis skip-r-def upower-rep-eq)

```

update-uexpr-rep-eq-thms

```

lemma Sup-power-expand:
  fixes P :: nat  $\Rightarrow$  'a::complete-lattice
  shows  $P(0) \sqcap (\bigsqcap i. P(i+1)) = (\bigsqcap i. P(i))$ 
proof -
  have UNIV = insert (0::nat) {1..}
    by auto
  moreover have  $(\bigsqcap i. P(i)) = \bigsqcap (P \restriction UNIV)$ 
    by (blast)
  moreover have  $\bigsqcap (P \restriction insert\ 0\ \{1..\}) = P(0) \sqcap SUPREMUM\ \{1..\}\ P$ 
    by (simp)
  moreover have  $SUPREMUM\ \{1..\}\ P = (\bigsqcap i. P(i+1))$ 
    by (simp add: atLeast-Suc-greaterThan greaterThan-0)
  ultimately show ?thesis
    by (simp only:)
qed

```

```

lemma Sup-upto-Suc:  $(\bigsqcap i \in \{0..Suc\ n\}. P \wedge i) = (\bigsqcap i \in \{0..n\}. P \wedge i) \sqcap P \wedge Suc\ n$ 
proof -
  have  $(\bigsqcap i \in \{0..Suc\ n\}. P \wedge i) = (\bigsqcap i \in insert\ (Suc\ n)\ \{0..n\}. P \wedge i)$ 
    by (simp add: atLeast0-atMost-Suc)
  also have  $\dots = P \wedge Suc\ n \sqcap (\bigsqcap i \in \{0..n\}. P \wedge i)$ 
    by (simp)
  finally show ?thesis
    by (simp add: Lattices.sup-commute)
qed

```

The following two proofs are adapted from the AFP entry [Kleene Algebra](#). See also [2, 1].

```

lemma upower-inductl:  $Q \sqsubseteq ((P ;; Q) \sqcap R) \Longrightarrow Q \sqsubseteq P \wedge n ;; R$ 
proof (induct n)
  case 0
  then show ?case by (auto)
next
  case (Suc n)
  then show ?case
    by (auto simp add: upred-semiring.power-Suc, metis (no-types, hide-lams) dual-order.trans order-refl
      segr-assoc segr-mono)

```

qed

**lemma** *upower-inductr*:  
 assumes  $Q \sqsubseteq R \sqcap (Q ;; P)$   
 shows  $Q \sqsubseteq R ;; (P \wedge n)$   
**using** *assms* **proof** (*induct*  $n$ )  
 case 0  
 then show ?case **by** *auto*  
**next**  
 case (*Suc*  $n$ )  
 have  $R ;; P \wedge \text{Suc } n = (R ;; P \wedge n) ;; P$   
**by** (*metis* *segr-assoc upred-semiring.power-Suc2*)  
 also have  $Q ;; P \sqsubseteq \dots$   
**by** (*meson* *Suc.hyps assms eq-iff segr-mono*)  
 also have  $Q \sqsubseteq \dots$   
**using** *assms* **by** *auto*  
 finally show ?case .  
 qed

**lemma** *SUP-atLeastAtMost-first*:  
 fixes  $P :: \text{nat} \Rightarrow 'a::\text{complete-lattice}$   
 assumes  $m \leq n$   
 shows  $(\bigsqcap_{i \in \{m..n\}}. P(i)) = P(m) \sqcap (\bigsqcap_{i \in \{\text{Suc } m..n\}}. P(i))$   
**by** (*metis* *SUP-insert assms atLeastAtMost-insertL*)

**lemma** *upower-segr-iter*:  $P \wedge n = (;; Q : \text{replicate } n P \cdot Q)$   
**by** (*induct*  $n$ , *simp-all* *add: upred-semiring.power-Suc*)

**lemma** *assigns-power*:  $\langle f \rangle_a \wedge n = \langle f \wedge n \rangle_a$   
**by** (*induct*  $n$ , *rel-auto+*)

## 18.14 Kleene Star

**definition** *ustar* ::  $'\alpha \text{ hrel} \Rightarrow ' \alpha \text{ hrel} (-^* [999] 999)$  **where**  
 $P^* = (\bigsqcap_{i \in \{0..\}} \cdot P^i)$

**lemma** *ustar-rep-eq*:  
 $\llbracket P^* \rrbracket_e = (\lambda b. b \in (\{p. \llbracket P \rrbracket_e p\}^*))$   
**by** (*simp* *add: ustar-def, rel-auto, simp-all* *add: relpow-imp-rtrancl rtrancl-imp-relpow*)

**update-uexpr-rep-eq-thms**

## 18.15 Kleene Plus

**purge-notation** *trancl*  $((-^+) [1000] 999)$

**definition** *uplus* ::  $'\alpha \text{ hrel} \Rightarrow ' \alpha \text{ hrel} (-^+ [999] 999)$  **where**  
*[upred-defs]*:  $P^+ = P ;; P^*$

**lemma** *uplus-power-def*:  $P^+ = (\bigsqcap i \cdot P \wedge (\text{Suc } i))$   
**by** (*simp* *add: uplus-def ustar-def seq-UNF-distl' UNF-atLeast-Suc upred-semiring.power-Suc*)

## 18.16 Omega

**definition** *uomega* ::  $'\alpha \text{ hrel} \Rightarrow ' \alpha \text{ hrel} (-^\omega [999] 999)$  **where**  
 $P^\omega = (\mu X \cdot P \cdot X ;; X)$

### 18.17 Relation Algebra Laws

**theorem** *RA1*:  $(P ;; (Q ;; R)) = ((P ;; Q) ;; R)$   
 by (*simp add: seqr-assoc*)

**theorem** *RA2*:  $(P ;; II) = P$   $(II ;; P) = P$   
 by *simp-all*

**theorem** *RA3*:  $P^{--} = P$   
 by *simp*

**theorem** *RA4*:  $(P ;; Q)^- = (Q^- ;; P^-)$   
 by *simp*

**theorem** *RA5*:  $(P \vee Q)^- = (P^- \vee Q^-)$   
 by (*rel-auto*)

**theorem** *RA6*:  $((P \vee Q) ;; R) = (P ;; R \vee Q ;; R)$   
 using *seqr-or-distl* by *blast*

**theorem** *RA7*:  $((P^- ;; (\neg(P ;; Q))) \vee (\neg Q)) = (\neg Q)$   
 by (*rel-auto*)

### 18.18 Kleene Algebra Laws

**lemma** *ustar-alt-def*:  $P^* = (\bigcap i. P \cdot P^* \cdot i)$   
 by (*simp add: ustar-def*)

**theorem** *ustar-sub-unfoldl*:  $P^* \sqsubseteq II \sqcap (P ;; P^*)$   
 by (*rel-simp, simp add: rtrancl-into-trancl2 trancl-into-rtrancl*)

**theorem** *ustar-inductl*:  
 assumes  $Q \sqsubseteq R$   $Q \sqsubseteq P ;; Q$   
 shows  $Q \sqsubseteq P^* ;; R$

**proof** –  
 have  $P^* ;; R = (\bigcap i. P \cdot P^* \cdot i ;; R)$   
 by (*simp add: ustar-def UINF-as-Sup-collect' seq-SUP-distr*)  
 also have  $Q \sqsubseteq \dots$   
 by (*simp add: SUP-least assms upower-inductl*)  
 finally show *?thesis* .  
**qed**

**theorem** *ustar-inductr*:  
 assumes  $Q \sqsubseteq R$   $Q \sqsubseteq Q ;; P$   
 shows  $Q \sqsubseteq R ;; P^*$

**proof** –  
 have  $R ;; P^* = (\bigcap i. R ;; P \cdot P^* \cdot i)$   
 by (*simp add: ustar-def UINF-as-Sup-collect' seq-SUP-distl*)  
 also have  $Q \sqsubseteq \dots$   
 by (*simp add: SUP-least assms upower-inductr*)  
 finally show *?thesis* .  
**qed**

**lemma** *ustar-refines-nu*:  $(\nu X. P \cdot (P ;; X) \sqcap II) \sqsubseteq P^*$   
 by (*metis (no-types, lifting) lfp-greatest semilattice-sup-class.le-sup-iff semilattice-sup-class.sup-idem upred-semiring.mult-2-right*)

*upred-semiring.one-add-one ustar-inductl)*

**lemma** *ustar-as-nu*:  $P^* = (\nu X \cdot (P ;; X) \sqcap II)$   
**proof** (*rule antisym*)  
 show  $(\nu X \cdot (P ;; X) \sqcap II) \sqsubseteq P^*$   
 by (*simp add: ustar-refines-nu*)  
 show  $P^* \sqsubseteq (\nu X \cdot (P ;; X) \sqcap II)$   
 by (*metis lfp-lowerbound upred-semiring.add-commute ustar-sub-unfoldl*)  
**qed**

**lemma** *ustar-unfoldl*:  $P^* = II \sqcap (P ;; P^*)$   
**apply** (*simp add: ustar-as-nu*)  
**apply** (*subst lfp-unfold*)  
**apply** (*rule monoI*)  
**apply** (*rel-auto*)  
**done**

While loop can be expressed using Kleene star

**lemma** *while-star-form*:  
 $\text{while } b \text{ do } P \text{ od} = (P \triangleleft b \triangleright_r II)^* ;; [(\neg b)]^\top$   
**proof** –  
 have 1: *Continuous*  $(\lambda X. P ;; X \triangleleft b \triangleright_r II)$   
 by (*rel-auto*)  
 have *while*  $b \text{ do } P \text{ od} = (\bigsqcap i. ((\lambda X. P ;; X \triangleleft b \triangleright_r II) \hat{\ } i) \text{ false})$   
 by (*simp add: 1 false-upred-def sup-continuous-Continuous sup-continuous-lfp while-top-def*)  
 also have ... =  $((\lambda X. P ;; X \triangleleft b \triangleright_r II) \hat{\ } 0) \text{ false} \sqcap (\bigsqcap i. ((\lambda X. P ;; X \triangleleft b \triangleright_r II) \hat{\ } (i+1)) \text{ false})$   
 by (*subst Sup-power-expand, simp*)  
 also have ... =  $(\bigsqcap i. ((\lambda X. P ;; X \triangleleft b \triangleright_r II) \hat{\ } (i+1)) \text{ false})$   
 by (*simp*)  
 also have ... =  $(\bigsqcap i. (P \triangleleft b \triangleright_r II) \hat{\ } i ;; (\text{false} \triangleleft b \triangleright_r II))$   
**proof** (*rule SUP-cong, simp-all*)  
 fix *i*  
 show  $P ;; ((\lambda X. P ;; X \triangleleft b \triangleright_r II) \hat{\ } i) \text{ false} \triangleleft b \triangleright_r II = (P \triangleleft b \triangleright_r II) \hat{\ } i ;; (\text{false} \triangleleft b \triangleright_r II)$   
**proof** (*induct i*)  
 case 0  
 then show ?case by *simp*  
**next**  
 case (*Suc i*)  
 then show ?case  
 by (*simp add: upred-semiring.power-Suc*)  
 (*metis (no-types, lifting) RA1 comp-cond-left-distr cond-L6 upred-semiring.mult.left-neutral*)  
**qed**  
**qed**  
 also have ... =  $(\bigsqcap i \in \{0..\} \cdot (P \triangleleft b \triangleright_r II) \hat{\ } i ;; [(\neg b)]^\top)$   
 by (*rel-auto*)  
 also have ... =  $(P \triangleleft b \triangleright_r II)^* ;; [(\neg b)]^\top$   
 by (*metis seq-UNF-distr ustar-def*)  
 finally show ?thesis .  
**qed**

## 18.19 Omega Algebra Laws

**lemma** *uomega-induct*:  
 $P ;; P^\omega \sqsubseteq P^\omega$   
 by (*simp add: uomega-def, metis eq-refl gfp-unfold monoI seqr-mono*)

## 18.20 Refinement Laws

**lemma** *skip-r-refine*:

$(p \Rightarrow p) \sqsubseteq II$   
**by** *pred-blast*

**lemma** *conj-refine-left*:

$(Q \Rightarrow P) \sqsubseteq R \Longrightarrow P \sqsubseteq (Q \wedge R)$   
**by** (*rel-auto*)

**lemma** *pre-weak-rel*:

**assumes** ' $Pre \Rightarrow I$ '  
**and**  $(I \Rightarrow Post) \sqsubseteq P$   
**shows**  $(Pre \Rightarrow Post) \sqsubseteq P$   
**using** *assms* **by** (*rel-auto*)

**lemma** *cond-refine-rel*:

**assumes**  $S \sqsubseteq ([b]_{<} \wedge P)$   $S \sqsubseteq ([\neg b]_{<} \wedge Q)$   
**shows**  $S \sqsubseteq P \triangleleft b \triangleright_r Q$   
**by** (*metis aext-not assms(1) assms(2) cond-def lift-rcond-def utp-pred-laws.le-sup-iff*)

**lemma** *seq-refine-pred*:

**assumes**  $([b]_{<} \Rightarrow [s]_{>}) \sqsubseteq P$  **and**  $([s]_{<} \Rightarrow [c]_{>}) \sqsubseteq Q$   
**shows**  $([b]_{<} \Rightarrow [c]_{>}) \sqsubseteq (P ;; Q)$   
**using** *assms* **by** *rel-auto*

**lemma** *seq-refine-unrest*:

**assumes**  $out\alpha \nmid b$   $in\alpha \nmid c$   
**assumes**  $(b \Rightarrow [s]_{>}) \sqsubseteq P$  **and**  $([s]_{<} \Rightarrow c) \sqsubseteq Q$   
**shows**  $(b \Rightarrow c) \sqsubseteq (P ;; Q)$   
**using** *assms* **by** *rel-blast*

## 18.21 Domain and Range Laws

**lemma** *Dom-conv-Ran*:

$Dom(P^-) = Ran(P)$   
**by** (*rel-auto*)

**lemma** *Ran-conv-Dom*:

$Ran(P^-) = Dom(P)$   
**by** (*rel-auto*)

**lemma** *Dom-skip*:

$Dom(II) = true$   
**by** (*rel-auto*)

**lemma** *Dom-assigns*:

$Dom(\langle \sigma \rangle_a) = true$   
**by** (*rel-auto*)

**lemma** *Dom-miracle*:

$Dom(false) = false$   
**by** (*rel-auto*)

**lemma** *Dom-assume*:

$Dom([b]^\top) = b$

by (rel-auto)

**lemma** *Dom-seq*:

$Dom(P ;; Q) = Dom(P ;; [Dom(Q)]^\top)$

by (rel-auto)

**lemma** *Dom-disj*:

$Dom(P \vee Q) = (Dom(P) \vee Dom(Q))$

by (rel-auto)

**lemma** *Dom-inf*:

$Dom(P \sqcap Q) = (Dom(P) \vee Dom(Q))$

by (rel-auto)

**lemma** *assume-Dom*:

$[Dom(P)]^\top ;; P = P$

by (rel-auto)

end

## 19 UTP Theories

**theory** *utp-theory*

**imports** *utp-rel-laws*

**begin**

Here, we mechanise a representation of UTP theories using locales [4]. We also link them to the HOL-Algebra library [5], which allows us to import properties from complete lattices and Galois connections.

### 19.1 Complete lattice of predicates

**definition** *upred-lattice* :: ( $'\alpha$  upred) gorder ( $\mathcal{P}$ ) **where**

*upred-lattice* = ( $\lambda$  carrier = UNIV, eq = (=), le = ( $\sqsubseteq$ ))

$\mathcal{P}$  is the complete lattice of alphabetised predicates. All other theories will be defined relative to it.

**interpretation** *upred-lattice*: complete-lattice  $\mathcal{P}$

**proof** (unfold-locales, simp-all add: upred-lattice-def)

fix  $A :: '\alpha$  upred set

show  $\exists s. \text{is-lub } (\lambda \text{ carrier} = \text{UNIV}, \text{eq} = (=), \text{le} = (\sqsubseteq)) s A$

apply (rule-tac  $x = \bigsqcup A$  in exI)

apply (rule least-UpperI)

apply (auto intro: Inf-greatest simp add: Inf-lower Upper-def)

done

show  $\exists i. \text{is-glb } (\lambda \text{ carrier} = \text{UNIV}, \text{eq} = (=), \text{le} = (\sqsubseteq)) i A$

apply (rule-tac  $x = \bigsqcap A$  in exI)

apply (rule greatest-LowerI)

apply (auto intro: Sup-least simp add: Sup-upper Lower-def)

done

qed

**lemma** *upred-weak-complete-lattice* [simp]: weak-complete-lattice  $\mathcal{P}$

by (simp add: upred-lattice.weak.weak-complete-lattice-axioms)

**lemma** *upred-lattice-eq* [simp]:  
 $(\cdot =_{\mathcal{P}}) = (=)$   
**by** (simp add: upred-lattice-def)

**lemma** *upred-lattice-le* [simp]:  
 $le_{\mathcal{P}} P Q = (P \sqsubseteq Q)$   
**by** (simp add: upred-lattice-def)

**lemma** *upred-lattice-carrier* [simp]:  
 $carrier \mathcal{P} = UNIV$   
**by** (simp add: upred-lattice-def)

**lemma** *Healthy-fixed-points* [simp]:  $fps \mathcal{P} H = \llbracket H \rrbracket_H$   
**by** (simp add: fps-def upred-lattice-def Healthy-def)

**lemma** *upred-lattice-Idempotent* [simp]:  $Idem_{\mathcal{P}} H = Idempotent H$   
**using** upred-lattice.weak-partial-order-axioms **by** (auto simp add: idempotent-def Idempotent-def)

**lemma** *upred-lattice-Monotonic* [simp]:  $Monop_{\mathcal{P}} H = Monotonic H$   
**using** upred-lattice.weak-partial-order-axioms **by** (auto simp add: isotone-def mono-def)

## 19.2 UTP theories hierarchy

**definition** *utp-order* ::  $('α \times 'α) \text{ health} \Rightarrow 'α \text{ hrel gorder}$  **where**  
 $utp\text{-order } H = (\mid carrier = \{P. P \text{ is } H\}, eq = (=), le = (\sqsubseteq) \mid)$

Constant *utp-order* obtains the order structure associated with a UTP theory. Its carrier is the set of healthy predicates, equality is HOL equality, and the order is refinement.

**lemma** *utp-order-carrier* [simp]:  
 $carrier (utp\text{-order } H) = \llbracket H \rrbracket_H$   
**by** (simp add: utp-order-def)

**lemma** *utp-order-eq* [simp]:  
 $eq (utp\text{-order } T) = (=)$   
**by** (simp add: utp-order-def)

**lemma** *utp-order-le* [simp]:  
 $le (utp\text{-order } T) = (\sqsubseteq)$   
**by** (simp add: utp-order-def)

**lemma** *utp-partial-order*:  $partial\text{-order } (utp\text{-order } T)$   
**by** (unfold-locales, simp-all add: utp-order-def)

**lemma** *utp-weak-partial-order*:  $weak\text{-partial-order } (utp\text{-order } T)$   
**by** (unfold-locales, simp-all add: utp-order-def)

**lemma** *mono-Monotone-utp-order*:  
 $mono f \Longrightarrow Monotone (utp\text{-order } T) f$   
**apply** (auto simp add: isotone-def)  
**apply** (metis partial-order-def utp-partial-order)  
**apply** (metis monoD)  
**done**

**lemma** *isotone-utp-orderI*:  $Monotonic H \Longrightarrow isotone (utp\text{-order } X) (utp\text{-order } Y) H$

**by** (*auto simp add: mono-def isotone-def utp-weak-partial-order*)

**lemma** *Mono-utp-orderI*:

$\llbracket \bigwedge P Q. \llbracket P \sqsubseteq Q; P \text{ is } H; Q \text{ is } H \rrbracket \implies F(P) \sqsubseteq F(Q) \rrbracket \implies \text{Mono}_{\text{utp-order } H} F$   
**by** (*auto simp add: isotone-def utp-weak-partial-order*)

The UTP order can equivalently be characterised as the fixed point lattice, *fpl*.

**lemma** *utp-order-fpl*:  $\text{utp-order } H = \text{fpl } \mathcal{P} H$

**by** (*auto simp add: utp-order-def upred-lattice-def fps-def Healthy-def*)

### 19.3 UTP theory hierarchy

We next define a hierarchy of locales that characterise different classes of UTP theory. Minimally we require that a UTP theory's healthiness condition is idempotent.

**locale** *utp-theory* =

**fixes** *hcond* ::  $'\alpha \text{ hrel} \Rightarrow '\alpha \text{ hrel } (\mathcal{H})$

**assumes** *HCond-Idem*:  $\mathcal{H}(\mathcal{H}(P)) = \mathcal{H}(P)$

**begin**

**abbreviation** *thy-order* ::  $'\alpha \text{ hrel } \text{gorder}$  **where**

*thy-order*  $\equiv \text{utp-order } \mathcal{H}$

**lemma** *HCond-Idempotent [closure,intro]*: *Idempotent*  $\mathcal{H}$

**by** (*simp add: Idempotent-def HCond-Idem*)

**sublocale** *utp-po*: *partial-order utp-order*  $\mathcal{H}$

**by** (*unfold-locales, simp-all add: utp-order-def*)

We need to remove some transitivity rules to stop them being applied in calculations

**declare** *utp-po.trans* [*trans del*]

**end**

**locale** *utp-theory-lattice* = *utp-theory* +

**assumes** *uthy-lattice*: *complete-lattice* (*utp-order*  $\mathcal{H}$ )

**begin**

**sublocale** *complete-lattice utp-order*  $\mathcal{H}$

**by** (*simp add: uthy-lattice*)

**declare** *top-closed* [*simp del*]

**declare** *bottom-closed* [*simp del*]

The healthiness conditions of a UTP theory lattice form a complete lattice, and allows us to make use of complete lattice results from HOL-Algebra [5], such as the Knaster-Tarski theorem. We can also retrieve lattice operators as below.

**abbreviation** *utp-top* ( $\top$ )

**where** *utp-top*  $\equiv \text{top } (\text{utp-order } \mathcal{H})$

**abbreviation** *utp-bottom* ( $\perp$ )

**where** *utp-bottom*  $\equiv \text{bottom } (\text{utp-order } \mathcal{H})$

**abbreviation** *utp-join* (**infixl**  $\sqcup$  65) **where**

*utp-join*  $\equiv \text{join } (\text{utp-order } \mathcal{H})$



**abbreviation** *utp-meet* (**infixl**  $\sqcap$  70) **where**  
*utp-meet*  $\equiv$  *meet* (*utp-order*  $\mathcal{H}$ )

**abbreviation** *utp-sup* (**infixr**  $\sqcup$  90) **where**  
*utp-sup*  $\equiv$  *Lattice.sup* (*utp-order*  $\mathcal{H}$ )

**abbreviation** *utp-inf* (**infixr**  $\sqcap$  90) **where**  
*utp-inf*  $\equiv$  *Lattice.inf* (*utp-order*  $\mathcal{H}$ )

**abbreviation** *utp-gfp* ( $\nu$ ) **where**  
*utp-gfp*  $\equiv$  *GREATEST-FP* (*utp-order*  $\mathcal{H}$ )

**abbreviation** *utp-lfp* ( $\mu$ ) **where**  
*utp-lfp*  $\equiv$  *LEAST-FP* (*utp-order*  $\mathcal{H}$ )

**end**

**syntax**

*-tmu* :: *logic*  $\Rightarrow$  *pttrn*  $\Rightarrow$  *logic*  $\Rightarrow$  *logic* ( $\mu_1$  -  $\cdot$  -  $[0, 10]$  10)  
*-tnu* :: *logic*  $\Rightarrow$  *pttrn*  $\Rightarrow$  *logic*  $\Rightarrow$  *logic* ( $\nu_1$  -  $\cdot$  -  $[0, 10]$  10)

**notation** *gfp* ( $\mu$ )

**notation** *lfp* ( $\nu$ )

**translations**

$\mu_H X \cdot P == \text{CONST LEAST-FP } (\text{CONST utp-order } H) (\lambda X. P)$   
 $\nu_H X \cdot P == \text{CONST GREATEST-FP } (\text{CONST utp-order } H) (\lambda X. P)$

**lemma** *upred-lattice-inf*:

*Lattice.inf*  $\mathcal{P} A = \sqcap A$

**by** (*metis Sup-least Sup-upper UNIV-I antisym-conv subsetI upred-lattice.weak.inf-greatest upred-lattice.weak.inf-lower upred-lattice-carrier upred-lattice-le*)

We can then derive a number of properties about these operators, as below.

**context** *utp-theory-lattice*

**begin**

**lemma** *LFP-healthy-comp*:  $\mu F = \mu (F \circ \mathcal{H})$

**proof** –

**have**  $\{P. (P \text{ is } \mathcal{H}) \wedge F P \sqsubseteq P\} = \{P. (P \text{ is } \mathcal{H}) \wedge F (\mathcal{H} P) \sqsubseteq P\}$

**by** (*auto simp add: Healthy-def*)

**thus** *?thesis*

**by** (*simp add: LEAST-FP-def*)

**qed**

**lemma** *GFP-healthy-comp*:  $\nu F = \nu (F \circ \mathcal{H})$

**proof** –

**have**  $\{P. (P \text{ is } \mathcal{H}) \wedge P \sqsubseteq F P\} = \{P. (P \text{ is } \mathcal{H}) \wedge P \sqsubseteq F (\mathcal{H} P)\}$

**by** (*auto simp add: Healthy-def*)

**thus** *?thesis*

**by** (*simp add: GREATEST-FP-def*)

**qed**

**lemma** *top-healthy [closure]*:  $\top \text{ is } \mathcal{H}$

```

using weak.top-closed by auto

lemma bottom-healthy [closure]:  $\perp$  is  $\mathcal{H}$ 
using weak.bottom-closed by auto

lemma utp-top:  $P$  is  $\mathcal{H} \implies P \sqsubseteq \top$ 
using weak.top-higher by auto

lemma utp-bottom:  $P$  is  $\mathcal{H} \implies \perp \sqsubseteq P$ 
using weak.bottom-lower by auto

end

lemma upred-top:  $\top_{\mathcal{P}} = \text{false}$ 
using ball-UNIV greatest-def by fastforce

lemma upred-bottom:  $\perp_{\mathcal{P}} = \text{true}$ 
by fastforce

```

One way of obtaining a complete lattice is showing that the healthiness conditions are monotone, which the below locale characterises.

```

locale utp-theory-mono = utp-theory +
assumes HCond-Mono [closure,intro]: Monotonic  $\mathcal{H}$ 

sublocale utp-theory-mono  $\subseteq$  utp-theory-lattice
proof –
interpret weak-complete-lattice  $\text{fpl } \mathcal{P} \mathcal{H}$ 
by (rule Knaster-Tarski, auto)

have complete-lattice ( $\text{fpl } \mathcal{P} \mathcal{H}$ )
by (unfold-locales, simp add: fps-def sup-exists, (blast intro: sup-exists inf-exists)+)

hence complete-lattice ( $\text{utp-order } \mathcal{H}$ )
by (simp add: utp-order-def, simp add: upred-lattice-def)

thus utp-theory-lattice  $\mathcal{H}$ 
by (simp add: utp-theory-axioms utp-theory-lattice.intro utp-theory-lattice-axioms.intro)
qed

```

In a monotone theory, the top and bottom can always be obtained by applying the healthiness condition to the predicate top and bottom, respectively.

```

context utp-theory-mono
begin

lemma healthy-top:  $\top = \mathcal{H}(\text{false})$ 
proof –
have  $\top = \top_{\text{fpl } \mathcal{P} \mathcal{H}}$ 
by (simp add: utp-order-fpl)
also have  $\dots = \mathcal{H} \top_{\mathcal{P}}$ 
using Knaster-Tarski-idem-extremes(1)[of  $\mathcal{P} \mathcal{H}$ ]
by (simp add: HCond-Idempotent HCond-Mono)
also have  $\dots = \mathcal{H} \text{ false}$ 
by (simp add: upred-top)
finally show ?thesis .
qed

```

```

lemma healthy-bottom:  $\perp = \mathcal{H}(\text{true})$ 
proof -
  have  $\perp = \perp_{fpl} \mathcal{P} \mathcal{H}$ 
    by (simp add: utp-order-fpl)
  also have  $\dots = \mathcal{H} \perp_{\mathcal{P}}$ 
    using Knaster-Tarski-idem-extremes(2)[of  $\mathcal{P} \mathcal{H}$ ]
    by (simp add: HCond-Idempotent HCond-Mono)
  also have  $\dots = \mathcal{H} \text{ true}$ 
    by (simp add: upred-bottom)
  finally show ?thesis .
qed

lemma healthy-inf:
  assumes  $A \subseteq \llbracket \mathcal{H} \rrbracket_H$ 
  shows  $\sqcap A = \mathcal{H} (\sqcap A)$ 
  using Knaster-Tarski-idem-inf-eq[OF upred-weak-complete-lattice, of  $\mathcal{H}$ ]
  by (simp, metis HCond-Idempotent HCond-Mono assms partial-object.simps(3) upred-lattice-def upred-lattice-inf
  utp-order-def)

end

locale utp-theory-continuous = utp-theory +
  assumes HCond-Cont [closure,intro]: Continuous  $\mathcal{H}$ 

sublocale utp-theory-continuous  $\subseteq$  utp-theory-mono
proof
  show Monotonic  $\mathcal{H}$ 
    by (simp add: Continuous-Monotonic HCond-Cont)
qed

context utp-theory-continuous
begin

lemma healthy-inf-cont:
  assumes  $A \subseteq \llbracket \mathcal{H} \rrbracket_H$   $A \neq \{\}$ 
  shows  $\sqcap A = \sqcap A$ 
proof -
  have  $\sqcap A = \sqcap (\mathcal{H}'A)$ 
    using Continuous-def HCond-Cont assms(1) assms(2) healthy-inf by auto
  also have  $\dots = \sqcap A$ 
    by (unfold Healthy-carrier-image[OF assms(1)], simp)
  finally show ?thesis .
qed

lemma healthy-inf-def:
  assumes  $A \subseteq \llbracket \mathcal{H} \rrbracket_H$ 
  shows  $\sqcap A = (\text{if } (A = \{\}) \text{ then } \top \text{ else } (\sqcap A))$ 
  using assms healthy-inf-cont weak.weak-inf-empty by auto

lemma healthy-meet-cont:
  assumes  $P$  is  $\mathcal{H}$   $Q$  is  $\mathcal{H}$ 
  shows  $P \sqcap Q = P \sqcap Q$ 
  using healthy-inf-cont[of  $\{P, Q\}$ ] assms
  by (simp add: Healthy-if-meet-def)

```

**lemma** *meet-is-healthy* [closure]:  
**assumes**  $P$  is  $\mathcal{H}$   $Q$  is  $\mathcal{H}$   
**shows**  $P \sqcap Q$  is  $\mathcal{H}$   
**by** (*metis Continuous-Disjunctous Disjunctuous-def HCond-Cont Healthy-def' assms(1) assms(2)*)

**lemma** *meet-bottom* [simp]:  
**assumes**  $P$  is  $\mathcal{H}$   
**shows**  $P \sqcap \perp = \perp$   
**by** (*simp add: assms semilattice-sup-class.sup-absorb2 utp-bottom*)

**lemma** *meet-top* [simp]:  
**assumes**  $P$  is  $\mathcal{H}$   
**shows**  $P \sqcap \top = P$   
**by** (*simp add: assms semilattice-sup-class.sup-absorb1 utp-top*)

The UTP theory lfp operator can be rewritten to the alphabetised predicate lfp when in a continuous context.

**theorem** *utp-lfp-def*:  
**assumes** *Monotonic*  $F$   $F \in \llbracket \mathcal{H} \rrbracket_H \rightarrow \llbracket \mathcal{H} \rrbracket_H$   
**shows**  $\mu F = (\mu X \cdot F(\mathcal{H}(X)))$   
**proof** (*rule antisym*)  
**have** *ne*:  $\{P. (P \text{ is } \mathcal{H}) \wedge F P \sqsubseteq P\} \neq \{\}$   
**proof** –  
**have**  $F \top \sqsubseteq \top$   
**using** *assms(2) utp-top weak.top-closed* **by** *force*  
**thus** *?thesis*  
**by** (*auto, rule-tac x=* $\top$  *in exI, auto simp add: top-healthy*)  
**qed**  
**show**  $\mu F \sqsubseteq (\mu X \cdot F(\mathcal{H} X))$   
**proof** –  
**have**  $\sqcap \{P. (P \text{ is } \mathcal{H}) \wedge F(P) \sqsubseteq P\} \sqsubseteq \sqcap \{P. F(\mathcal{H}(P)) \sqsubseteq P\}$   
**proof** –  
**have**  $1: \bigwedge P. F(\mathcal{H}(P)) = \mathcal{H}(F(\mathcal{H}(P)))$   
**by** (*metis HCond-Idem Healthy-def assms(2) funcset-mem mem-Collect-eq*)  
**show** *?thesis*  
**proof** (*rule Sup-least, auto*)  
**fix**  $P$   
**assume**  $a: F(\mathcal{H} P) \sqsubseteq P$   
**hence**  $F: (F(\mathcal{H} P)) \sqsubseteq (\mathcal{H} P)$   
**by** (*metis 1 HCond-Mono mono-def*)  
**show**  $\sqcap \{P. (P \text{ is } \mathcal{H}) \wedge F P \sqsubseteq P\} \sqsubseteq P$   
**proof** (*rule Sup-upper2[of F (* $\mathcal{H} P$ *)]*)  
**show**  $F(\mathcal{H} P) \in \{P. (P \text{ is } \mathcal{H}) \wedge F P \sqsubseteq P\}$   
**proof** (*auto*)  
**show**  $F(\mathcal{H} P)$  is  $\mathcal{H}$   
**by** (*metis 1 Healthy-def*)  
**show**  $F(F(\mathcal{H} P)) \sqsubseteq F(\mathcal{H} P)$   
**using** *F mono-def assms(1)* **by** *blast*  
**qed**  
**show**  $F(\mathcal{H} P) \sqsubseteq P$   
**by** (*simp add: a*)  
**qed**  
**qed**  
**qed**

```

    with ne show ?thesis
    by (simp add: LEAST-FP-def gfp-def, subst healthy-inf-cont, auto simp add: lfp-def)
qed
from ne show  $(\mu X \cdot F(\mathcal{H} X)) \sqsubseteq \mu F$ 
  apply (simp add: LEAST-FP-def gfp-def, subst healthy-inf-cont, auto simp add: lfp-def)
  apply (rule Sup-least)
  apply (auto simp add: Healthy-def Sup-upper)
done
qed

lemma UINF-ind-Healthy [closure]:
  assumes  $\bigwedge i. P(i) \text{ is } \mathcal{H}$ 
  shows  $(\bigcap i \cdot P(i)) \text{ is } \mathcal{H}$ 
  by (simp add: closure assms)

```

end

In another direction, we can also characterise UTP theories that are relational. Minimally this requires that the healthiness condition is closed under sequential composition.

```

locale utp-theory-rel =
  utp-theory +
  assumes Healthy-Sequence [closure]:  $\llbracket P \text{ is } \mathcal{H}; Q \text{ is } \mathcal{H} \rrbracket \implies (P ;; Q) \text{ is } \mathcal{H}$ 
begin

```

```

lemma upower-Suc-Healthy [closure]:
  assumes  $P \text{ is } \mathcal{H}$ 
  shows  $P \wedge \text{Suc } n \text{ is } \mathcal{H}$ 
  by (induct n, simp-all add: closure assms upred-semiring.power-Suc)

```

end

```

locale utp-theory-cont-rel = utp-theory-rel + utp-theory-continuous
begin

```

```

lemma seq-cont-Sup-distl:
  assumes  $P \text{ is } \mathcal{H} \ A \subseteq \llbracket \mathcal{H} \rrbracket_H \ A \neq \{\}$ 
  shows  $P ;; (\bigcap A) = \bigcap \{P ;; Q \mid Q \in A\}$ 
proof -
  have  $\{P ;; Q \mid Q \in A\} \subseteq \llbracket \mathcal{H} \rrbracket_H$ 
    using Healthy-Sequence assms(1) assms(2) by (auto)
  thus ?thesis
    by (simp add: healthy-inf-cont seq-Sup-distl setcompr-eq-image assms)
qed

```

```

lemma seq-cont-Sup-distr:
  assumes  $Q \text{ is } \mathcal{H} \ A \subseteq \llbracket \mathcal{H} \rrbracket_H \ A \neq \{\}$ 
  shows  $(\bigcap A) ;; Q = \bigcap \{P ;; Q \mid P \in A\}$ 
proof -
  have  $\{P ;; Q \mid P \in A\} \subseteq \llbracket \mathcal{H} \rrbracket_H$ 
    using Healthy-Sequence assms(1) assms(2) by (auto)
  thus ?thesis
    by (simp add: healthy-inf-cont seq-Sup-distr setcompr-eq-image assms)
qed

```

```

lemma uplus-healthy [closure]:
  assumes  $P$  is  $\mathcal{H}$ 
  shows  $P^+$  is  $\mathcal{H}$ 
  by (simp add: uplus-power-def closure assms)

```

**end**

There also exist UTP theories with units. Not all theories have both a left and a right unit (e.g. H1-H2 designs) and so we split up the locale into two cases.

```

locale utp-theory-units =
  utp-theory-rel +
  fixes utp-unit ( $\mathcal{II}$ )
  assumes Healthy-Unit [closure]:  $\mathcal{II}$  is  $\mathcal{H}$ 
begin

```

We can characterise the theory Kleene star by lifting the relational one.

```

definition utp-star ( $-\star$  [999] 999) where
[upred-defs]: utp-star  $P = (P^* ;; \mathcal{II})$ 

```

We can then characterise tests as refinements of units.

```

definition utp-test :: ' $a$  hrel  $\Rightarrow$  bool' where
[upred-defs]: utp-test  $b = (\mathcal{II} \sqsubseteq b)$ 

```

**end**

```

locale utp-theory-left-unital =
  utp-theory-units +
  assumes Unit-Left:  $P$  is  $\mathcal{H} \Longrightarrow (\mathcal{II} ;; P) = P$ 

```

```

locale utp-theory-right-unital =
  utp-theory-units +
  assumes Unit-Right:  $P$  is  $\mathcal{H} \Longrightarrow (P ;; \mathcal{II}) = P$ 

```

```

locale utp-theory-unital =
  utp-theory-left-unital + utp-theory-right-unital
begin

```

```

lemma Unit-self [simp]:
   $\mathcal{II} ;; \mathcal{II} = \mathcal{II}$ 
  by (simp add: Healthy-Unit Unit-Right)

```

```

lemma utest-intro:
   $\mathcal{II} \sqsubseteq P \Longrightarrow \text{utp-test } P$ 
  by (simp add: utp-test-def)

```

```

lemma utest-Unit [closure]:
  utp-test  $\mathcal{II}$ 
  by (simp add: utp-test-def)

```

**end**

```

locale utp-theory-mono-unital = utp-theory-unital + utp-theory-mono
begin

```

```

lemma utest-Top [closure]: utp-test  $\top$ 

```

```

    by (simp add: Healthy-Unit utp-test-def utp-top)

end

locale utp-theory-cont-unital = utp-theory-cont-rel + utp-theory-unital

sublocale utp-theory-cont-unital  $\subseteq$  utp-theory-mono-unital
  by (simp add: utp-theory-mono-axioms utp-theory-mono-unital-def utp-theory-unital-axioms)

locale utp-theory-unital-zero =
  utp-theory-unital +
  utp-theory-lattice +
  assumes Top-Left-Zero:  $P \text{ is } \mathcal{H} \implies \top ;; P = \top$ 

locale utp-theory-cont-unital-zero =
  utp-theory-cont-unital + utp-theory-unital-zero
begin

lemma Top-test-Right-Zero:
  assumes  $b \text{ is } \mathcal{H} \text{ utp-test } b$ 
  shows  $b ;; \top = \top$ 
proof -
  have  $b \sqcap \mathcal{I} = \mathcal{I}$ 
  by (meson assms(2) semilattice-sup-class.le-iff-sup utp-test-def)
  then show ?thesis
  by (metis (no-types) Top-Left-Zero Unit-Left assms(1) meet-top top-healthy upred-semiring.distrib-right)
qed

end

```

## 19.4 Theory of relations

```

interpretation rel-theory: utp-theory-mono-unital id skip-r
  rewrites rel-theory.utp-top = false
  and rel-theory.utp-bottom = true
  and carrier (utp-order id) = UNIV
  and ( $P \text{ is id}$ ) = True
proof -
  show utp-theory-mono-unital id II
  by (unfold-locales, simp-all add: Healthy-def)
  then interpret utp-theory-mono-unital id skip-r
  by simp
  show utp-top = false utp-bottom = true
  by (simp-all add: healthy-top healthy-bottom)
  show carrier (utp-order id) = UNIV ( $P \text{ is id}$ ) = True
  by (auto simp add: utp-order-def Healthy-def)
qed

```

**thm** *rel-theory.GFP-unfold*

## 19.5 Theory links

We can also describe links between theories, such a Galois connections and retractions, using the following notation.

**definition** *mk-conn*  $(- \Leftarrow \langle -, - \rangle \Rightarrow - [90, 0, 0, 91] \ 91)$  **where**

$H1 \Leftarrow \langle \mathcal{H}_1, \mathcal{H}_2 \rangle \Rightarrow H2 \equiv (\mid \text{orderA} = \text{utp-order } H1, \text{orderB} = \text{utp-order } H2, \text{lower} = \mathcal{H}_2, \text{upper} = \mathcal{H}_1 \mid)$

**lemma** *mk-conn-orderA* [simp]:  $\mathcal{X}_{H1} \Leftarrow \langle \mathcal{H}_1, \mathcal{H}_2 \rangle \Rightarrow H2 = \text{utp-order } H1$   
**by** (simp add: mk-conn-def)

**lemma** *mk-conn-orderB* [simp]:  $\mathcal{Y}_{H1} \Leftarrow \langle \mathcal{H}_1, \mathcal{H}_2 \rangle \Rightarrow H2 = \text{utp-order } H2$   
**by** (simp add: mk-conn-def)

**lemma** *mk-conn-lower* [simp]:  $\pi_* H1 \Leftarrow \langle \mathcal{H}_1, \mathcal{H}_2 \rangle \Rightarrow H2 = \mathcal{H}_1$   
**by** (simp add: mk-conn-def)

**lemma** *mk-conn-upper* [simp]:  $\pi^* H1 \Leftarrow \langle \mathcal{H}_1, \mathcal{H}_2 \rangle \Rightarrow H2 = \mathcal{H}_2$   
**by** (simp add: mk-conn-def)

**lemma** *galois-comp*:  $(H2 \Leftarrow \langle \mathcal{H}_3, \mathcal{H}_4 \rangle \Rightarrow H3) \circ_g (H1 \Leftarrow \langle \mathcal{H}_1, \mathcal{H}_2 \rangle \Rightarrow H2) = H1 \Leftarrow \langle \mathcal{H}_1 \circ \mathcal{H}_3, \mathcal{H}_4 \circ \mathcal{H}_2 \rangle \Rightarrow H3$   
**by** (simp add: comp-galconn-def mk-conn-def)

Example Galois connection / retract: Existential quantification

**lemma** *Idempotent-ex*:  $\text{mwb-lens } x \Longrightarrow \text{Idempotent } (ex \ x)$   
**by** (simp add: Idempotent-def exists-twice)

**lemma** *Monotonic-ex*:  $\text{mwb-lens } x \Longrightarrow \text{Monotonic } (ex \ x)$   
**by** (simp add: mono-def ex-mono)

**lemma** *ex-closed-unrest*:  
 $\text{vwb-lens } x \Longrightarrow \llbracket ex \ x \rrbracket_H = \{P. x \# P\}$   
**by** (simp add: Healthy-def unrest-as-exists)

Any theory can be composed with an existential quantification to produce a Galois connection

**theorem** *ex-retract*:

**assumes** *vwb-lens*  $x$  *Idempotent*  $H$   $ex \ x \circ H = H \circ ex \ x$

**shows** *retract*  $((ex \ x \circ H) \Leftarrow \langle ex \ x, H \rangle \Rightarrow H)$

**proof** (*unfold-locales*, *simp-all*)

**show**  $H \in \llbracket ex \ x \circ H \rrbracket_H \rightarrow \llbracket H \rrbracket_H$

**using** *Healthy-Idempotent* *assms* **by** *blast*

**from** *assms*(1) *assms*(3)[*THEN sym*] **show**  $ex \ x \in \llbracket H \rrbracket_H \rightarrow \llbracket ex \ x \circ H \rrbracket_H$

**by** (simp add: *Pi-iff Healthy-def fun-eq-iff exists-twice*)

**fix**  $P \ Q$

**assume**  $P$  *is*  $(ex \ x \circ H)$   $Q$  *is*  $H$

**thus**  $(H \ P \sqsubseteq Q) = (P \sqsubseteq (\exists \ x \cdot Q))$

**by** (*metis* (*no-types*, *lifting*) *Healthy-Idempotent Healthy-if assms comp-apply dual-order.trans ex-weakens utp-pred-laws.ex-mono vwb-lens-wb*)

**next**

**fix**  $P$

**assume**  $P$  *is*  $(ex \ x \circ H)$

**thus**  $(\exists \ x \cdot H \ P) \sqsubseteq P$

**by** (simp add: *Healthy-def*)

**qed**

**corollary** *ex-retract-id*:

**assumes** *vwb-lens*  $x$

**shows** *retract*  $(ex \ x \Leftarrow \langle ex \ x, id \rangle \Rightarrow id)$

**using** *assms* *ex-retract*[**where**  $H=id$ ] **by** (*auto*)

**end**



## 20 Relational Hoare calculus

```

theory utp-hoare
  imports
    utp-rel-laws
    utp-theory
begin

```

### 20.1 Hoare Triple Definitions and Tactics

**definition** *hoare-r* :: ' $\alpha \text{ cond} \Rightarrow \alpha \text{ hrel} \Rightarrow \alpha \text{ cond} \Rightarrow \text{bool}$ ' ( $\{\cdot\}/\cdot/\{\cdot\}_u$ ) **where**  
 $\{p\}Q\{r\}_u = (([p]_{<} \Rightarrow [r]_{>}) \sqsubseteq Q)$

**declare** *hoare-r-def* [*upred-defs*]

**named-theorems** *hoare* **and** *hoare-safe*

**method** *hoare-split* **uses** *hr* =  
 ((*simp add: assigns-comp*)?, — Combine Assignments where possible  
 (*auto*  
   *intro: hoare intro!: hoare-safe hr*  
   *simp add: conj-comm conj-assoc usubst unrest*))[1] — Apply Hoare logic laws

**method** *hoare-auto* **uses** *hr* = (*hoare-split hr: hr; (rel-simp)?, auto?*)

### 20.2 Basic Laws

**lemma** *hoare-meaning*:  
 $\{P\}S\{Q\}_u = (\forall s s'. \llbracket P \rrbracket_e s \wedge \llbracket S \rrbracket_e (s, s') \longrightarrow \llbracket Q \rrbracket_e s')$   
**by** (*rel-auto*)

**lemma** *hoare-assume*:  $\{P\}S\{Q\}_u \Longrightarrow ?[P] ;; S = ?[P] ;; S ;; ?[Q]$   
**by** (*rel-auto*)

**lemma** *hoare-r-conj* [*hoare-safe*]:  $\llbracket \{p\}Q\{r\}_u; \{p\}Q\{s\}_u \rrbracket \Longrightarrow \{p\}Q\{r \wedge s\}_u$   
**by** *rel-auto*

**lemma** *hoare-r-weaken-pre* [*hoare*]:  
 $\{p\}Q\{r\}_u \Longrightarrow \{p \wedge q\}Q\{r\}_u$   
 $\{q\}Q\{r\}_u \Longrightarrow \{p \wedge q\}Q\{r\}_u$   
**by** *rel-auto+*

**lemma** *pre-str-hoare-r*:  
**assumes** ' $p_1 \Rightarrow p_2$ ' **and**  $\{p_2\}C\{q\}_u$   
**shows**  $\{p_1\}C\{q\}_u$   
**using** *assms* **by** *rel-auto*

**lemma** *post-weak-hoare-r*:  
**assumes**  $\{p\}C\{q_2\}_u$  **and** ' $q_2 \Rightarrow q_1$ '  
**shows**  $\{p\}C\{q_1\}_u$   
**using** *assms* **by** *rel-auto*

**lemma** *hoare-r-conseq*:  $\llbracket 'p_1 \Rightarrow p_2'; \{p_2\}S\{q_2\}_u; 'q_2 \Rightarrow q_1' \rrbracket \Longrightarrow \{p_1\}S\{q_1\}_u$   
**by** *rel-auto*

## 20.3 Assignment Laws

**lemma** *assigns-hoare-r* [*hoare-safe*]:  $'p \Rightarrow \sigma \dagger q' \Longrightarrow \{p\}\langle\sigma\rangle_a\{q\}_u$   
**by** *rel-auto*

**lemma** *assigns-backward-hoare-r*:  
 $\{\sigma \dagger p\}\langle\sigma\rangle_a\{p\}_u$   
**by** *rel-auto*

**lemma** *assign-floyd-hoare-r*:  
**assumes** *vwb-lens x*  
**shows**  $\{p\} \text{ assign-r } x \ e \ \{\exists v \cdot p[\ll v \gg/x] \wedge \&x =_u e[\ll v \gg/x]\}_u$   
**using** *assms*  
**by** (*rel-auto, metis vwb-lens-wb wb-lens.get-put*)

**lemma** *assigns-init-hoare* [*hoare-safe*]:  
 $\ll \text{vwb-lens } x; x \# p; x \# v; \&x =_u v \wedge p\{S\{q\}_u\} \rrbracket \Longrightarrow \{p\}x := v ;; S\{q\}_u$   
**by** (*rel-auto*)

**lemma** *skip-hoare-r* [*hoare-safe*]:  $\{p\}II\{p\}_u$   
**by** *rel-auto*

**lemma** *skip-hoare-impl-r* [*hoare-safe*]:  $'p \Rightarrow q' \Longrightarrow \{p\}II\{q\}_u$   
**by** *rel-auto*

## 20.4 Sequence Laws

**lemma** *seq-hoare-r*:  $\ll \{p\}Q_1\{s\}_u ; \{s\}Q_2\{r\}_u \rrbracket \Longrightarrow \{p\}Q_1 ;; Q_2\{r\}_u$   
**by** *rel-auto*

**lemma** *seq-hoare-invariant* [*hoare-safe*]:  $\ll \{p\}Q_1\{p\}_u ; \{p\}Q_2\{p\}_u \rrbracket \Longrightarrow \{p\}Q_1 ;; Q_2\{p\}_u$   
**by** *rel-auto*

**lemma** *seq-hoare-stronger-pre-1* [*hoare-safe*]:  
 $\ll \{p \wedge q\}Q_1\{p \wedge q\}_u ; \{p \wedge q\}Q_2\{q\}_u \rrbracket \Longrightarrow \{p \wedge q\}Q_1 ;; Q_2\{q\}_u$   
**by** *rel-auto*

**lemma** *seq-hoare-stronger-pre-2* [*hoare-safe*]:  
 $\ll \{p \wedge q\}Q_1\{p \wedge q\}_u ; \{p \wedge q\}Q_2\{p\}_u \rrbracket \Longrightarrow \{p \wedge q\}Q_1 ;; Q_2\{p\}_u$   
**by** *rel-auto*

**lemma** *seq-hoare-inv-r-2* [*hoare*]:  $\ll \{p\}Q_1\{q\}_u ; \{q\}Q_2\{q\}_u \rrbracket \Longrightarrow \{p\}Q_1 ;; Q_2\{q\}_u$   
**by** *rel-auto*

**lemma** *seq-hoare-inv-r-3* [*hoare*]:  $\ll \{p\}Q_1\{p\}_u ; \{p\}Q_2\{q\}_u \rrbracket \Longrightarrow \{p\}Q_1 ;; Q_2\{q\}_u$   
**by** *rel-auto*

## 20.5 Conditional Laws

**lemma** *cond-hoare-r* [*hoare-safe*]:  $\ll \{b \wedge p\}S\{q\}_u ; \{\neg b \wedge p\}T\{q\}_u \rrbracket \Longrightarrow \{p\}S \triangleleft b \triangleright_r T\{q\}_u$   
**by** *rel-auto*

**lemma** *cond-hoare-r-wp*:  
**assumes**  $\{p'\}S\{q\}_u$  **and**  $\{p''\}T\{q\}_u$   
**shows**  $\{(b \wedge p') \vee (\neg b \wedge p'')\}S \triangleleft b \triangleright_r T\{q\}_u$   
**using** *assms* **by** *pred-simp*

**lemma** *cond-hoare-r-sp*:  
**assumes**  $\langle \{b \wedge p\} S \{q\}_u \rangle$  **and**  $\langle \{\neg b \wedge p\} T \{s\}_u \rangle$   
**shows**  $\langle \{p\} S \triangleleft b \triangleright_r T \{q \vee s\}_u \rangle$   
**using** *assms* **by** *pred-simp*

## 20.6 Recursion Laws

**lemma** *nu-hoare-r-partial*:  
**assumes** *induct-step*:  
 $\bigwedge st P. \{p\} P \{q\}_u \implies \{p\} F P \{q\}_u$   
**shows**  $\{p\} \nu F \{q\}_u$   
**by** (*meson hoare-r-def induct-step lfp-lowerbound order-refl*)

**lemma** *mu-hoare-r*:  
**assumes** *WF*: *wf* *R*  
**assumes** *M*:*mono* *F*  
**assumes** *induct-step*:  
 $\bigwedge st P. \{p \wedge (e, \ll st \gg)_u \in_u \ll R \gg\} P \{q\}_u \implies \{p \wedge e =_u \ll st \gg\} F P \{q\}_u$   
**shows**  $\{p\} \mu F \{q\}_u$   
**unfolding** *hoare-r-def*  
**proof** (*rule mu-rec-total-utp-rule*[*OF* *WF* *M* , *of* - *e* ], *goal-cases*)  
**case** (*1 st*)  
**then show** *?case*  
**using** *induct-step*[*unfolded hoare-r-def*, *of* ( $\lceil p \rceil < \wedge (\lceil e \rceil <, \ll st \gg)_u \in_u \ll R \gg \Rightarrow \lceil q \rceil >$ ) *st*]  
**by** (*simp add: alpha*)  
**qed**

**lemma** *mu-hoare-r'*:  
**assumes** *WF*: *wf* *R*  
**assumes** *M*:*mono* *F*  
**assumes** *induct-step*:  
 $\bigwedge st P. \{p \wedge (e, \ll st \gg)_u \in_u \ll R \gg\} P \{q\}_u \implies \{p \wedge e =_u \ll st \gg\} F P \{q\}_u$   
**assumes** *I0*: '*p*'  $\Rightarrow$  *p*'  
**shows**  $\{p'\} \mu F \{q\}_u$   
**by** (*meson I0 M WF induct-step mu-hoare-r pre-str-hoare-r*)

## 20.7 Iteration Rules

**lemma** *iter-hoare-r*:  $\{P\} S \{P\}_u \implies \{P\} S^* \{P\}_u$   
**by** (*rel-simp'*, *metis* (*mono-tags*, *lifting*) *mem-Collect-eq rtrancl-induct*)

**lemma** *while-hoare-r* [*hoare-safe*]:  
**assumes**  $\{p \wedge b\} S \{p\}_u$   
**shows**  $\{p\} \text{while } b \text{ do } S \text{ od } \{\neg b \wedge p\}_u$   
**using** *assms*  
**by** (*simp add: while-top-def hoare-r-def, rule-tac lfp-lowerbound*) (*rel-auto*)

**lemma** *while-invr-hoare-r* [*hoare-safe*]:  
**assumes**  $\{p \wedge b\} S \{p\}_u$  '*pre*'  $\Rightarrow$  *p*' ' $(\neg b \wedge p) \Rightarrow$  *post*'  
**shows**  $\{pre\} \text{while } b \text{ invr } p \text{ do } S \text{ od } \{post\}_u$   
**by** (*metis assms hoare-r-conseq while-hoare-r while-inv-def*)

**lemma** *while-r-minimal-partial*:  
**assumes** *seq-step*: '*p*'  $\Rightarrow$  *invar*'  
**assumes** *induct-step*:  $\{invar \wedge b\} C \{invar\}_u$

**shows**  $\llbracket p \rrbracket \text{while } b \text{ do } C \text{ od } \llbracket \neg b \wedge \text{invar} \rrbracket_u$   
**using** *induct-step pre-str-hoare-r seq-step while-hoare-r* **by** *blast*

**lemma** *approx-chain*:

$(\bigcap n::\text{nat}. \lceil p \wedge v <_u \ll n \gg \rceil_<) = \lceil p \rceil_<$   
**by** (*rel-auto*)

Total correctness law for Hoare logic, based on constructive chains. This is limited to variants that have natural numbers as their range.

**lemma** *while-term-hoare-r*:

**assumes**  $\bigwedge z::\text{nat}. \llbracket p \wedge b \wedge v =_u \ll z \gg \rrbracket S \llbracket p \wedge v <_u \ll z \gg \rrbracket_u$   
**shows**  $\llbracket p \rrbracket \text{while}_\perp b \text{ do } S \text{ od } \llbracket \neg b \wedge p \rrbracket_u$

**proof** –

**have**  $(\lceil p \rceil_< \Rightarrow \lceil \neg b \wedge p \rceil_>) \sqsubseteq (\mu X. S ;; X \triangleleft b \triangleright_r II)$

**proof** (*rule mu-refine-intro*)

**from** *assms* **show**  $(\lceil p \rceil_< \Rightarrow \lceil \neg b \wedge p \rceil_>) \sqsubseteq S ;; (\lceil p \rceil_< \Rightarrow \lceil \neg b \wedge p \rceil_>) \triangleleft b \triangleright_r II$   
**by** (*rel-auto*)

**let**  $?E = \lambda n. \lceil p \wedge v <_u \ll n \gg \rceil_<$

**show**  $(\lceil p \rceil_< \wedge (\mu X. S ;; X \triangleleft b \triangleright_r II)) = (\lceil p \rceil_< \wedge (\nu X. S ;; X \triangleleft b \triangleright_r II))$

**proof** (*rule constr-fp-uniq[where E=?E]*)

**show**  $(\bigcap n. ?E(n)) = \lceil p \rceil_<$   
**by** (*rel-auto*)

**show** *mono*  $(\lambda X. S ;; X \triangleleft b \triangleright_r II)$   
**by** (*simp add: cond-mono monoI seqr-mono*)

**show** *constr*  $(\lambda X. S ;; X \triangleleft b \triangleright_r II) ?E$   
**proof** (*rule constrI*)

**show** *chain*  $?E$

**proof** (*rule chainI*)

**show**  $\lceil p \wedge v <_u \ll 0 \gg \rceil_< = \text{false}$   
**by** (*rel-auto*)

**show**  $\bigwedge i. \lceil p \wedge v <_u \ll \text{Suc } i \gg \rceil_< \sqsubseteq \lceil p \wedge v <_u \ll i \gg \rceil_<$   
**by** (*rel-auto*)

**qed**

**from** *assms*

**show**  $\bigwedge X n. (S ;; X \triangleleft b \triangleright_r II \wedge \lceil p \wedge v <_u \ll n + 1 \gg \rceil_<) =$   
 $(S ;; (X \wedge \lceil p \wedge v <_u \ll n \gg \rceil_<) \triangleleft b \triangleright_r II \wedge \lceil p \wedge v <_u \ll n + 1 \gg \rceil_<)$

**apply** (*rel-auto*)

**using** *less-antisym less-trans* **apply** *blast*

**done**

**qed**

**qed**

**qed**

**thus** *?thesis*

**by** (*simp add: hoare-r-def while-bot-def*)

**qed**

**lemma** *while-vrt-hoare-r* [*hoare-safe*]:

**assumes**  $\bigwedge z::nat. \llbracket p \wedge b \wedge v =_u \llbracket z \rrbracket \rrbracket S \llbracket p \wedge v <_u \llbracket z \rrbracket \rrbracket_u 'pre \Rightarrow p' '(\neg b \wedge p) \Rightarrow post'$   
**shows**  $\llbracket pre \rrbracket while\ b\ invr\ p\ vrt\ v\ do\ S\ od\ \llbracket post \rrbracket_u$   
**apply** (rule hoare-r-conseq[OF assms(2) - assms(3)])  
**apply** (simp add: while-vrt-def)  
**apply** (rule while-term-hoare-r[where v=v, OF assms(1)])  
**done**

General total correctness law based on well-founded induction

**lemma** *while-wf-hoare-r*:

**assumes** *WF*: wf *R*  
**assumes** *I0*: ' $pre \Rightarrow p$ '  
**assumes** *induct-step*:  $\bigwedge st. \llbracket b \wedge p \wedge e =_u \llbracket st \rrbracket \rrbracket Q \llbracket p \wedge (e, \llbracket st \rrbracket)_u \in_u \llbracket R \rrbracket \rrbracket_u$   
**assumes** *PHI*: ' $(\neg b \wedge p) \Rightarrow post$ '  
**shows**  $\llbracket pre \rrbracket while_{\perp}\ b\ invr\ p\ do\ Q\ od\ \llbracket post \rrbracket_u$   
**unfolding** *hoare-r-def while-inv-bot-def while-bot-def*  
**proof** (rule pre-weak-rel[of -  $\lceil p \rceil_{<}$  ])  
**from** *I0* **show** ' $\lceil pre \rceil_{<} \Rightarrow \lceil p \rceil_{<}$ '  
**by** rel-auto  
**show** ( $\lceil p \rceil_{<} \Rightarrow \lceil post \rceil_{>}$ )  $\sqsubseteq$  ( $\mu X. X \cdot Q ;; X \triangleleft b \triangleright_r II$ )  
**proof** (rule mu-rec-total-utp-rule[where e=e, OF *WF*])  
**show** *Monotonic* ( $\lambda X. X ;; X \triangleleft b \triangleright_r II$ )  
**by** (simp add: closure)  
**have** *induct-step'*:  $\bigwedge st. (\llbracket b \wedge p \wedge e =_u \llbracket st \rrbracket \rrbracket_{<} \Rightarrow (\llbracket p \wedge (e, \llbracket st \rrbracket)_u \in_u \llbracket R \rrbracket \rrbracket_{>})) \sqsubseteq Q$   
**using** *induct-step* **by** rel-auto  
**with** *PHI*  
**show**  $\bigwedge st. (\lceil p \rceil_{<} \wedge \lceil e \rceil_{<} =_u \llbracket st \rrbracket \Rightarrow \lceil post \rceil_{>}) \sqsubseteq Q ;; (\lceil p \rceil_{<} \wedge (\lceil e \rceil_{<}, \llbracket st \rrbracket)_u \in_u \llbracket R \rrbracket \Rightarrow \lceil post \rceil_{>})$   
 $\triangleleft b \triangleright_r II$   
**by** (rel-auto)  
**qed**  
**qed**

## 20.8 Frame Rules

Frame rule: If starting *S* in a state satisfying *pestablishe**sq* in the final state, then we can insert an invariant predicate *r* when *S* is framed by *a*, provided that *r* does not refer to variables in the frame, and *q* does not refer to variables outside the frame.

**lemma** *frame-hoare-r*:

**assumes** *vwb-lens* *a a*  $\# r\ a \triangleleft q \llbracket p \rrbracket P \llbracket q \rrbracket_u$   
**shows**  $\llbracket p \wedge r \rrbracket a: [P] \llbracket q \wedge r \rrbracket_u$   
**using** *assms*  
**by** (rel-auto, metis)

**lemma** *frame-strong-hoare-r* [*hoare-safe*]:

**assumes** *vwb-lens* *a a*  $\# r\ a \triangleleft q \llbracket p \wedge r \rrbracket S \llbracket q \rrbracket_u$   
**shows**  $\llbracket p \wedge r \rrbracket a: [S] \llbracket q \wedge r \rrbracket_u$   
**using** *assms* **by** (rel-auto, metis)

**lemma** *frame-hoare-r'* [*hoare-safe*]:

**assumes** *vwb-lens* *a a*  $\# r\ a \triangleleft q \llbracket r \wedge p \rrbracket S \llbracket q \rrbracket_u$   
**shows**  $\llbracket r \wedge p \rrbracket a: [S] \llbracket r \wedge q \rrbracket_u$   
**using** *assms*  
**by** (simp add: frame-strong-hoare-r utp-pred-laws.inf commute)

**lemma** *antiframe-hoare-r*:

**assumes** *vwb-lens* *a a*  $\triangleleft r\ a \# q \llbracket p \rrbracket P \llbracket q \rrbracket_u$

**shows**  $\llbracket p \wedge r \rrbracket a : \llbracket P \rrbracket \llbracket q \wedge r \rrbracket_u$   
**using** *assms* **by** (*rel-auto*, *metis*)

**lemma** *antiframe-strong-hoare-r*:  
**assumes** *vwb-lens*  $a \Vdash r \ a \not\# q \ \llbracket p \wedge r \rrbracket P \llbracket q \rrbracket_u$   
**shows**  $\llbracket p \wedge r \rrbracket a : \llbracket P \rrbracket \llbracket q \wedge r \rrbracket_u$   
**using** *assms* **by** (*rel-auto*, *metis*)

**end**

## 21 Weakest (Liberal) Precondition Calculus

**theory** *utp-wp*  
**imports** *utp-hoare*  
**begin**

A very quick implementation of wlp – more laws still needed!

**named-theorems** *wp*

**method** *wp-tac* = (*simp add: wp*)

**consts**  
 $uwp :: 'a \Rightarrow 'b \Rightarrow 'c$

**syntax**  
 $-uwp \ :: \ logic \Rightarrow uexp \Rightarrow logic \ (\mathbf{infix} \ wp \ 60)$

**translations**  
 $-uwp \ P \ b \ == \ CONST \ uwp \ P \ b$

**definition**  $wp\text{-upred} :: ('\alpha, '\beta) \ urel \Rightarrow '\beta \ cond \Rightarrow '\alpha \ cond \ \mathbf{where}$   
 $wp\text{-upred} \ Q \ r = \lfloor \neg (Q \ ; \ ; \ (\neg \lceil r \rceil_{<})) :: ('\alpha, '\beta) \ urel \rfloor_{<}$

**adhoc-overloading**  
 $uwp \ wp\text{-upred}$

**declare**  $wp\text{-upred-def} \ [\textit{urel-defs}]$

**lemma**  $wp\text{-true} \ [wp]: p \ wp \ true = true$   
**by** (*rel-simp*)

**theorem**  $wp\text{-assigns-r} \ [wp]:$   
 $\langle \sigma \rangle_a \ wp \ r = \sigma \upharpoonright r$   
**by** *rel-auto*

**theorem**  $wp\text{-skip-r} \ [wp]:$   
 $\textit{II} \ wp \ r = r$   
**by** *rel-auto*

**theorem**  $wp\text{-abort} \ [wp]:$   
 $r \neq true \Longrightarrow true \ wp \ r = false$   
**by** *rel-auto*

**theorem**  $wp\text{-conj} \ [wp]:$   
 $P \ wp \ (q \wedge r) = (P \ wp \ q \wedge P \ wp \ r)$

by *rel-auto*

**theorem** *wp-seq-r* [*wp*]:  $(P \mathrel{;;} Q) \text{ wp } r = P \text{ wp } (Q \text{ wp } r)$   
by *rel-auto*

**theorem** *wp-choice* [*wp*]:  $(P \sqcap Q) \text{ wp } R = (P \text{ wp } R \wedge Q \text{ wp } R)$   
by (*rel-auto*)

**theorem** *wp-cond* [*wp*]:  $(P \triangleleft b \triangleright_r Q) \text{ wp } r = ((b \Rightarrow P \text{ wp } r) \wedge ((\neg b) \Rightarrow Q \text{ wp } r))$   
by *rel-auto*

**lemma** *wp-USUP-pre* [*wp*]:  $P \text{ wp } (\bigsqcup_{i \in \{0..n\}} Q(i)) = (\bigsqcup_{i \in \{0..n\}} P \text{ wp } Q(i))$   
by (*rel-auto*)

**theorem** *wp-hoare-link*:  
 $\{p\} Q \{r\}_u \longleftrightarrow (Q \text{ wp } r \sqsubseteq p)$   
by *rel-auto*

If two programs have the same weakest precondition for any postcondition then the programs are the same.

**theorem** *wp-eq-intro*:  $\llbracket \bigwedge r. P \text{ wp } r = Q \text{ wp } r \rrbracket \Longrightarrow P = Q$   
by (*rel-auto robust, fastforce+*)  
end

## 22 Dynamic Logic

**theory** *utp-dynlog*  
**imports** *utp-sequent utp-wp*  
**begin**

### 22.1 Definitions

**named-theorems** *dynlog-simp* and *dynlog-intro*

**definition** *dBox* ::  $'s \text{ hrel} \Rightarrow 's \text{ upred} \Rightarrow 's \text{ upred} \rightarrow ([\cdot] - [0,999] \ 999)$   
**where** [*upred-defs*]:  $dBox \ A \ \Phi = A \text{ wp } \Phi$

**definition** *dDia* ::  $'s \text{ hrel} \Rightarrow 's \text{ upred} \Rightarrow 's \text{ upred} \rightarrow (<->- [0,999] \ 999)$   
**where** [*upred-defs*]:  $dDia \ A \ \Phi = (\neg [A] \ (\neg \Phi))$

### 22.2 Box Laws

**lemma** *dBox-false* [*dynlog-simp*]:  $[false]\Phi = true$   
by (*rel-auto*)

**lemma** *dBox-skip* [*dynlog-simp*]:  $[I]\Phi = \Phi$   
by (*rel-auto*)

**lemma** *dBox-assigns* [*dynlog-simp*]:  $[\langle \sigma \rangle_a]\Phi = (\sigma \dagger \Phi)$   
by (*simp add: dBox-def wp-assigns-r*)

**lemma** *dBox-choice* [*dynlog-simp*]:  $[P \sqcap Q]\Phi = ([P]\Phi \wedge [Q]\Phi)$   
by (*rel-auto*)

**lemma** *dBox-seq*:  $[P \mathrel{;;} Q]\Phi = [P][Q]\Phi$

by (simp add: dBox-def wp-seq-r)

**lemma** dBox-star-unfold:  $[P^*]\Phi = (\Phi \wedge [P][P^*]\Phi)$   
 by (metis dBox-choice dBox-seq dBox-skip ustar-unfoldl)

**lemma** dBox-star-induct:  $'(\Phi \wedge [P^*](\Phi \Rightarrow [P]\Phi)) \Rightarrow [P^*]\Phi'$   
 by (rel-simp, metis (mono-tags, lifting) mem-Collect-eq rtrancl-induct)

**lemma** dBox-test:  $[?p]\Phi = (p \Rightarrow \Phi)$   
 by (rel-auto)

## 22.3 Diamond Laws

**lemma** dDia-false [dynlog-simp]:  $\langle \text{false} \rangle \Phi = \text{false}$   
 by (simp add: dBox-false dDia-def)

**lemma** dDia-skip [dynlog-simp]:  $\langle I \rangle \Phi = \Phi$   
 by (simp add: dBox-skip dDia-def)

**lemma** dDia-assigns [dynlog-simp]:  $\langle \langle \sigma \rangle_a \rangle \Phi = (\sigma \dagger \Phi)$   
 by (simp add: dBox-assigns dDia-def subst-not)

**lemma** dDia-choice:  $\langle P \sqcap Q \rangle \Phi = (\langle P \rangle \Phi \vee \langle Q \rangle \Phi)$   
 by (simp add: dBox-def dDia-def wp-choice)

**lemma** dDia-seq:  $\langle P ;; Q \rangle \Phi = \langle P \rangle \langle Q \rangle \Phi$   
 by (simp add: dBox-def dDia-def wp-seq-r)

**lemma** dDia-test:  $\langle ?p \rangle \Phi = (p \wedge \Phi)$   
 by (rel-auto)

## 22.4 Sequent Laws

**lemma** sBoxSeq [dynlog-simp]:  $\Gamma \Vdash [P ;; Q]\Phi \equiv \Gamma \Vdash [P][Q]\Phi$   
 by (simp add: dBox-def wp-seq-r)

**lemma** sBoxTest [dynlog-intro]:  $\Gamma \Vdash (b \Rightarrow \Psi) \Longrightarrow \Gamma \Vdash [?b]\Psi$   
 by (rel-auto)

**lemma** sBoxAssignFwd [dynlog-simp]:  $\llbracket \text{vwb-lens } x; x \# v; x \# \Gamma \rrbracket \Longrightarrow (\Gamma \Vdash [x := v]\Phi) = ((\&x =_u v \wedge \Gamma) \Vdash \Phi)$   
 by (rel-auto, metis vwb-lens-wb wb-lens.get-put)

**lemma** sBoxIndStar:  $\Vdash [\Phi \Rightarrow [P]\Phi]_u \Longrightarrow \Phi \Vdash [P^*]\Phi$   
 by (rel-simp, metis (mono-tags, lifting) mem-Collect-eq rtrancl-induct)

**lemma** hoare-as-dynlog:  $\llbracket p \rrbracket Q \llbracket r \rrbracket_u = (p \Vdash [Q]r)$   
 by (rel-auto)

end

## 23 State Variable Declaration Parser

**theory** utp-state-parser  
**imports** utp-rel



**begin**

This theory sets up a parser for state blocks, as an alternative way of providing lenses to a predicate. A program with local variables can be represented by a predicate indexed by a tuple of lenses, where each lens represents a variable. These lenses must then be supplied with respect to a suitable state space. Instead of creating a type to represent this alphabet, we can create a product type for the state space, with an entry for each variable. Then each variable becomes a composition of the  $fst_L$  and  $snd_L$  lenses to index the correct position in the variable vector. We first creation a vacuous definition that will mark when an indexed predicate denotes a state block.

**definition** *state-block* :: ( $'v \Rightarrow 'p$ )  $\Rightarrow 'v \Rightarrow 'p$  **where**  
 $[upred-defs]: state-block\ f\ x = f\ x$

We declare a number of syntax translations to produce lens and product types, to obtain a type for the overall state space, to construct a tuple that denotes the lens vector parameter, to construct the vector itself, and finally to construct the state declaration.

**syntax**

$-lensT :: type \Rightarrow type \Rightarrow type\ (LENSTYPE'(-, -'))$   
 $-pairT :: type \Rightarrow type \Rightarrow type\ (PAIRTYPE'(-, -'))$   
 $-state-type :: pttrn \Rightarrow type$   
 $-state-tuple :: type \Rightarrow pttrn \Rightarrow logic$   
 $-state-lenses :: pttrn \Rightarrow logic$   
 $-state-decl :: pttrn \Rightarrow logic \Rightarrow logic\ (LOCAL\ \cdot\ \cdot\ [0, 10]\ 10)$

**translations**

$(type)\ PAIRTYPE('a, 'b) \Rightarrow (type)\ 'a \times 'b$   
 $(type)\ LENTYPE('a, 'b) \Rightarrow (type)\ 'a \Longrightarrow 'b$   
  
 $-state-type\ (-constrain\ x\ t) \Rightarrow t$   
 $-state-type\ (CONST\ Pair\ (-constrain\ x\ t)\ vs) \Rightarrow -pairT\ t\ (-state-type\ vs)$   
  
 $-state-tuple\ st\ (-constrain\ x\ t) \Rightarrow -constrain\ x\ (-lensT\ t\ st)$   
 $-state-tuple\ st\ (CONST\ Pair\ (-constrain\ x\ t)\ vs) \Rightarrow$   
 $\quad CONST\ Product-Type.Pair\ (-constrain\ x\ (-lensT\ t\ st))\ (-state-tuple\ st\ vs)$   
  
 $-state-decl\ vs\ P \Rightarrow$   
 $\quad CONST\ state-block\ (-abs\ (-state-tuple\ (-state-type\ vs)\ vs)\ P)\ (-state-lenses\ vs)$   
 $-state-decl\ vs\ P \leq CONST\ state-block\ (-abs\ vs\ P)\ k$

**parse-translation** (

*let*  
 $open\ HOLogic;$   
 $val\ lens-comp = Const\ (@\{const-syntax\ lens-comp\}, dummyT);$   
 $val\ fst-lens = Const\ (@\{const-syntax\ fst-lens\}, dummyT);$   
 $val\ snd-lens = Const\ (@\{const-syntax\ snd-lens\}, dummyT);$   
 $val\ id-lens = Const\ (@\{const-syntax\ id-lens\}, dummyT);$   
 $(*\ Construct\ a\ tuple\ of\ lenses\ for\ each\ of\ the\ possible\ locally\ declared\ variables\ *)$   
*fun*  
 $state-lenses\ n\ st =$   
 $\quad if\ (n = 1)$   
 $\quad\quad then\ st$   
 $\quad\quad else\ pair-const\ dummyT\ dummyT\ \$\ (lens-comp\ \$\ fst-lens\ \$\ st)\ \$\ (state-lenses\ (n - 1)\ (lens-comp$   
 $\quad\quad \$\ snd-lens\ \$\ st));$   
*fun*

```

(* Add up the number of variable declarations in the tuple *)
var-decl-num (Const (@{const-syntax Product-Type.Pair},-) $ - $ vs) = var-decl-num vs + 1 |
var-decl-num - = 1;

```

```

  fun state-lens ctxt [vs] = state-lenses (var-decl-num vs) id-lens ;
in
[(-state-lenses, state-lens)]
end

```

### 23.1 Examples

```

term LOCAL (x::int, y::real, z::int) • x := (&x + &z)

```

```

lemma LOCAL p • II = II
  by (rel-auto)

```

**end**

## 24 Relational Operational Semantics

```

theory utp-rel-opsem
imports
  utp-rel-laws
  utp-hoare
begin

```

This theory uses the laws of relational calculus to create a basic operational semantics. It is based on Chapter 10 of the UTP book [22].

```

fun trel :: 'α usubst × 'α hrel ⇒ 'α usubst × 'α hrel ⇒ bool (infix →u 85) where
(σ, P) →u (ρ, Q) ⟷ ((⟨σ⟩a ;; P) ⊆ (⟨ρ⟩a ;; Q))

```

```

lemma trans-trel:
  ⟦ (σ, P) →u (ρ, Q); (ρ, Q) →u (φ, R) ⟧ ⟹ (σ, P) →u (φ, R)
by auto

```

```

lemma skip-trel: (σ, II) →u (σ, II)
by simp

```

```

lemma assigns-trel: (σ, ⟨ρ⟩a) →u (ρ ◦ σ, II)
by (simp add: assigns-comp)

```

```

lemma assign-trel:
  (σ, x := v) →u (σ(&x ↦s σ † v), II)
by (simp add: assigns-comp usubst)

```

```

lemma seq-trel:
  assumes (σ, P) →u (ρ, Q)
  shows (σ, P ;; R) →u (ρ, Q ;; R)
by (metis (no-types, lifting) assms order-refl seqr-assoc seqr-mono trel.simps)

```

```

lemma seq-skip-trel:
  (σ, II ;; P) →u (σ, P)

```

by *simp*

**lemma** *nondet-left-trel*:

$(\sigma, P \sqcap Q) \rightarrow_u (\sigma, P)$

by (*metis* (*no-types*, *hide-lams*) *disj-comm* *disj-upred-def* *semilattice-sup-class.sup.absorb-iff1* *semilattice-sup-class.sup.l* *seqr-or-distr* *trel.simps*)

**lemma** *nondet-right-trel*:

$(\sigma, P \sqcap Q) \rightarrow_u (\sigma, Q)$

by (*simp* *add*: *seqr-mono*)

**lemma** *rcond-true-trel*:

assumes  $\sigma \dagger b = \text{true}$

shows  $(\sigma, P \triangleleft b \triangleright_r Q) \rightarrow_u (\sigma, P)$

using *assms*

by (*simp* *add*: *assigns-r-comp* *usubst* *alpha* *cond-unit-T*)

**lemma** *rcond-false-trel*:

assumes  $\sigma \dagger b = \text{false}$

shows  $(\sigma, P \triangleleft b \triangleright_r Q) \rightarrow_u (\sigma, Q)$

using *assms*

by (*simp* *add*: *assigns-r-comp* *usubst* *alpha* *cond-unit-F*)

**lemma** *while-true-trel*:

assumes  $\sigma \dagger b = \text{true}$

shows  $(\sigma, \text{while } b \text{ do } P \text{ od}) \rightarrow_u (\sigma, P ;; \text{while } b \text{ do } P \text{ od})$

by (*metis* *assms* *rcond-true-trel* *while-unfold*)

**lemma** *while-false-trel*:

assumes  $\sigma \dagger b = \text{false}$

shows  $(\sigma, \text{while } b \text{ do } P \text{ od}) \rightarrow_u (\sigma, \text{II})$

by (*metis* *assms* *rcond-false-trel* *while-unfold*)

Theorem linking Hoare calculus and operational semantics. If we start  $Q$  in a state  $\sigma_0$  satisfying  $p$ , and  $Q$  reaches final state  $\sigma_1$  then  $r$  holds in this final state.

**theorem** *hoare-opsem-link*:

$\llbracket p \rrbracket Q \llbracket r \rrbracket_u = (\forall \sigma_0 \sigma_1. ' \sigma_0 \dagger p' \wedge (\sigma_0, Q) \rightarrow_u (\sigma_1, \text{II}) \longrightarrow ' \sigma_1 \dagger r')$

apply (*rel-auto*)

apply (*rename-tac*  $a \ b$ )

apply (*drule-tac*  $x=\lambda \cdot a \text{ in spec, simp}$ )

apply (*drule-tac*  $x=\lambda \cdot b \text{ in spec, simp}$ )

done

declare *trel.simps* [*simp del*]

end

## 25 Symbolic Evaluation of Relational Programs

**theory** *utp-sym-eval*

imports *utp-rel-opsem*

begin

The following operator applies a variable context  $\Gamma$  as an assignment, and composes it with a relation  $P$  for the purposes of evaluation.

**definition** *utp-sym-eval* :: 's usubst  $\Rightarrow$  's hrel  $\Rightarrow$  's hrel (**infixr**  $\models$  55) **where**  
*[upred-defs]: utp-sym-eval*  $\Gamma$   $P = (\langle \Gamma \rangle_a \;; P)$

**named-theorems** *symeval*

**lemma** *seq-symeval* [*symeval*]:  $\Gamma \models P \;; Q = (\Gamma \models P) \;; Q$   
**by** (*rel-auto*)

**lemma** *assigns-symeval* [*symeval*]:  $\Gamma \models \langle \sigma \rangle_a = (\sigma \circ \Gamma) \models II$   
**by** (*rel-auto*)

**lemma** *term-symeval* [*symeval*]:  $(\Gamma \models II) \;; P = \Gamma \models P$   
**by** (*rel-auto*)

**lemma** *if-true-symeval* [*symeval*]:  $\llbracket \Gamma \dagger b = \text{true} \rrbracket \Longrightarrow \Gamma \models (P \triangleleft b \triangleright_r Q) = \Gamma \models P$   
**by** (*simp add: utp-sym-eval-def usubst assigns-r-comp*)

**lemma** *if-false-symeval* [*symeval*]:  $\llbracket \Gamma \dagger b = \text{false} \rrbracket \Longrightarrow \Gamma \models (P \triangleleft b \triangleright_r Q) = \Gamma \models Q$   
**by** (*simp add: utp-sym-eval-def usubst assigns-r-comp*)

**lemma** *while-true-symeval* [*symeval*]:  $\llbracket \Gamma \dagger b = \text{true} \rrbracket \Longrightarrow \Gamma \models \text{while } b \text{ do } P \text{ od} = \Gamma \models (P \;; \text{while } b \text{ do } P \text{ od})$   
**by** (*subst while-unfold, simp add: symeval*)

**lemma** *while-false-symeval* [*symeval*]:  $\llbracket \Gamma \dagger b = \text{false} \rrbracket \Longrightarrow \Gamma \models \text{while } b \text{ do } P \text{ od} = \Gamma \models II$   
**by** (*subst while-unfold, simp add: symeval*)

**lemma** *while-inv-true-symeval* [*symeval*]:  $\llbracket \Gamma \dagger b = \text{true} \rrbracket \Longrightarrow \Gamma \models \text{while } b \text{ invr } S \text{ do } P \text{ od} = \Gamma \models (P \;; \text{while } b \text{ do } P \text{ od})$   
**by** (*metis while-inv-def while-true-symeval*)

**lemma** *while-inv-false-symeval* [*symeval*]:  $\llbracket \Gamma \dagger b = \text{false} \rrbracket \Longrightarrow \Gamma \models \text{while } b \text{ invr } S \text{ do } P \text{ od} = \Gamma \models II$   
**by** (*metis while-false-symeval while-inv-def*)

**method** *sym-eval* = (*simp add: symeval usubst lit-simps[THEN sym]*), (*simp del: One-nat-def add: One-nat-def[THEN sym]*)?

**syntax**

*-terminated* :: *logic*  $\Rightarrow$  *logic* (*terminated*: - [999] 999)

**translations**

*terminated*:  $\Gamma == \Gamma \models II$

**end**

## 26 Strong Postcondition Calculus

**theory** *utp-sp*  
**imports** *utp-wp*  
**begin**

**named-theorems** *sp*

**method** *sp-tac* = (*simp add: sp*)

**consts**

$usp :: 'a \Rightarrow 'b \Rightarrow 'c$  (**infix**  $sp$  60)

**definition**  $sp\text{-}upred :: 'a \text{ cond} \Rightarrow ('a, 'b) \text{ urel} \Rightarrow 'b \text{ cond}$  **where**  
 $sp\text{-}upred\ p\ Q = \lfloor ([p]_{>} ;; Q) :: ('a, 'b) \text{ urel} \rfloor_{>}$

**adhoc-overloading**

$usp\ sp\text{-}upred$

**declare**  $sp\text{-}upred\text{-}def$  [ $upred\text{-}defs$ ]

**lemma**  $sp\text{-}false$  [ $sp$ ]:  $p\ sp\ false = false$   
**by** ( $rel\text{-}simp$ )

**lemma**  $sp\text{-}true$  [ $sp$ ]:  $q \neq false \implies q\ sp\ true = true$   
**by** ( $rel\text{-}auto$ )

**lemma**  $sp\text{-}assigns\text{-}r$  [ $sp$ ]:  
 $vwb\text{-}lens\ x \implies (p\ sp\ x := e) = (\exists v \cdot p \llbracket \llbracket v \rrbracket / x \rrbracket \wedge \&x =_u e \llbracket \llbracket v \rrbracket / x \rrbracket)$   
**by** ( $rel\text{-}auto$ ,  $metis\ vwb\text{-}lens\text{-}wb\ wb\text{-}lens.get\text{-}put$ ,  $metis\ vwb\text{-}lens.put\text{-}eq$ )

**lemma**  $sp\text{-}it\text{-}is\text{-}post\text{-}condition$ :

$\{p\} C \{p\ sp\ C\}_u$   
**by**  $rel\text{-}blast$

**lemma**  $sp\text{-}it\text{-}is\text{-}the\text{-}strongest\text{-}post$ :

$\langle p\ sp\ C \Rightarrow Q \rangle \implies \{p\} C \{Q\}_u$   
**by**  $rel\text{-}blast$

**lemma**  $sp\text{-}so$ :

$\langle p\ sp\ C \Rightarrow Q \rangle = \{p\} C \{Q\}_u$   
**by**  $rel\text{-}blast$

**theorem**  $sp\text{-}hoare\text{-}link$ :

$\{p\} Q \{r\}_u \longleftrightarrow (r \sqsubseteq p\ sp\ Q)$   
**by**  $rel\text{-}auto$

**lemma**  $sp\text{-}while\text{-}r$  [ $sp$ ]:

**assumes**  $\langle pre \Rightarrow I \rangle$  **and**  $\langle \{I \wedge b\} C \{I\}_u \rangle$  **and**  $\langle I' \Rightarrow I \rangle$   
**shows**  $(pre\ sp\ \text{invar}\ I\ \text{while}_{\perp}\ b\ \text{do}\ C\ \text{od}) = (\neg b \wedge I)$   
**unfolding**  $sp\text{-}upred\text{-}def$   
**oops**

**theorem**  $sp\text{-}eq\text{-}intro$ :  $\llbracket \bigwedge r. r\ sp\ P = r\ sp\ Q \rrbracket \implies P = Q$

**by** ( $rel\text{-}auto\ robust$ ,  $fastforce+$ )

**lemma**  $wp\text{-}sp\text{-}sym$ :

$\langle prog\ wp\ (true\ sp\ prog) \rangle$   
**by**  $rel\text{-}auto$

**lemma**  $it\text{-}is\text{-}pre\text{-}condition$ :  $\{C\ wp\ Q\} C \{Q\}_u$

**by**  $rel\text{-}blast$

**lemma**  $it\text{-}is\text{-}the\text{-}weakest\text{-}pre$ :  $\langle P \Rightarrow C\ wp\ Q \rangle = \{P\} C \{Q\}_u$

**by**  $rel\text{-}blast$

**lemma** *s-pre*: ' $P \Rightarrow C \text{ wp } Q$ ' =  $\llbracket P \rrbracket C \llbracket Q \rrbracket_u$   
 by *rel-blast*

**end**

## 27 Concurrent Programming

**theory** *utp-concurrency*

**imports**

*utp-hoare*

*utp-rel*

*utp-tactics*

*utp-theory*

**begin**

In this theory we describe the UTP scheme for concurrency, *parallel-by-merge*, which provides a general parallel operator parametrised by a “merge predicate” that explains how to merge the after states of the composed predicates. It can thus be applied to many languages and concurrency schemes, with this theory providing a number of generic laws. The operator is explained in more detail in Chapter 7 of the UTP book [22].

### 27.1 Variable Renamings

In parallel-by-merge constructions, a merge predicate defines the behaviour following execution of of parallel processes,  $P \parallel Q$ , as a relation that merges the output of  $P$  and  $Q$ . In order to achieve this we need to separate the variable values output from  $P$  and  $Q$ , and in addition the variable values before execution. The following three constructs do these separations. The initial state-space before execution is  $'\alpha$ , the final state-space after the first parallel process is  $'\beta_0$ , and the final state-space for the second is  $'\beta_1$ . These three functions lift variables on these three state-spaces, respectively.

**alphabet**  $('\alpha, '\beta_0, '\beta_1) \text{ mrg} =$

*mrg-prior* ::  $'\alpha$

*mrg-left* ::  $'\beta_0$

*mrg-right* ::  $'\beta_1$

**definition** *pre-uvar* ::  $('a \Rightarrow '\alpha) \Rightarrow ('a \Rightarrow (''\alpha, '\beta_0, '\beta_1) \text{ mrg})$  **where**

[*upred-defs*]: *pre-uvar*  $x = x ;_L \text{ mrg-prior}$

**definition** *left-uvar* ::  $('a \Rightarrow '\beta_0) \Rightarrow ('a \Rightarrow (''\alpha, '\beta_0, '\beta_1) \text{ mrg})$  **where**

[*upred-defs*]: *left-uvar*  $x = x ;_L \text{ mrg-left}$

**definition** *right-uvar* ::  $('a \Rightarrow '\beta_1) \Rightarrow ('a \Rightarrow (''\alpha, '\beta_0, '\beta_1) \text{ mrg})$  **where**

[*upred-defs*]: *right-uvar*  $x = x ;_L \text{ mrg-right}$

We set up syntax for the three variable classes using a subscript  $<$ , 0- $x$ , and 1- $x$ , respectively.

**syntax**

*-svarpre* ::  $\text{svid} \Rightarrow \text{svid} \ (-< [995] \ 995)$

*-svarleft* ::  $\text{svid} \Rightarrow \text{svid} \ (0-- [995] \ 995)$

*-svarright* ::  $\text{svid} \Rightarrow \text{svid} \ (1-- [995] \ 995)$

**translations**

*-svarpre*  $x == \text{CONST } \text{pre-uvar } x$

$-svarleft\ x == CONST\ left-uvar\ x$   
 $-svarright\ x == CONST\ right-uvar\ x$   
 $-svarpre\ \Sigma \leq CONST\ pre-uvar\ 1_L$   
 $-svarleft\ \Sigma \leq CONST\ left-uvar\ 1_L$   
 $-svarright\ \Sigma \leq CONST\ right-uvar\ 1_L$

We proved behavedness closure properties about the lenses.

**lemma** *left-uvar* [simp]:  $vwb-lens\ x \implies vwb-lens\ (left-uvar\ x)$   
 by (simp add: *left-uvar-def*)

**lemma** *right-uvar* [simp]:  $vwb-lens\ x \implies vwb-lens\ (right-uvar\ x)$   
 by (simp add: *right-uvar-def*)

**lemma** *pre-uvar* [simp]:  $vwb-lens\ x \implies vwb-lens\ (pre-uvar\ x)$   
 by (simp add: *pre-uvar-def*)

**lemma** *left-uvar-mwb* [simp]:  $mwb-lens\ x \implies mwb-lens\ (left-uvar\ x)$   
 by (simp add: *left-uvar-def*)

**lemma** *right-uvar-mwb* [simp]:  $mwb-lens\ x \implies mwb-lens\ (right-uvar\ x)$   
 by (simp add: *right-uvar-def*)

**lemma** *pre-uvar-mwb* [simp]:  $mwb-lens\ x \implies mwb-lens\ (pre-uvar\ x)$   
 by (simp add: *pre-uvar-def*)

We prove various independence laws about the variable classes.

**lemma** *left-uvar-indep-right-uvar* [simp]:  
 $left-uvar\ x \bowtie right-uvar\ y$   
 by (simp add: *left-uvar-def right-uvar-def lens-comp-assoc[THEN sym]*)

**lemma** *left-uvar-indep-pre-uvar* [simp]:  
 $left-uvar\ x \bowtie pre-uvar\ y$   
 by (simp add: *left-uvar-def pre-uvar-def*)

**lemma** *left-uvar-indep-left-uvar* [simp]:  
 $x \bowtie y \implies left-uvar\ x \bowtie left-uvar\ y$   
 by (simp add: *left-uvar-def*)

**lemma** *right-uvar-indep-left-uvar* [simp]:  
 $right-uvar\ x \bowtie left-uvar\ y$   
 by (simp add: *lens-indep-sym*)

**lemma** *right-uvar-indep-pre-uvar* [simp]:  
 $right-uvar\ x \bowtie pre-uvar\ y$   
 by (simp add: *right-uvar-def pre-uvar-def*)

**lemma** *right-uvar-indep-right-uvar* [simp]:  
 $x \bowtie y \implies right-uvar\ x \bowtie right-uvar\ y$   
 by (simp add: *right-uvar-def*)

**lemma** *pre-uvar-indep-left-uvar* [simp]:  
 $pre-uvar\ x \bowtie left-uvar\ y$   
 by (simp add: *lens-indep-sym*)

**lemma** *pre-uvar-indep-right-uvar* [simp]:

*pre-uvar*  $x \bowtie$  *right-uvar*  $y$   
**by** (*simp* *add*: *lens-indep-sym*)

**lemma** *pre-uvar-indep-pre-uvar* [*simp*]:  
 $x \bowtie y \implies \text{pre-uvar } x \bowtie \text{pre-uvar } y$   
**by** (*simp* *add*: *pre-uvar-def*)

## 27.2 Merge Predicates

A merge predicate is a relation whose input has three parts: the prior variables, the output variables of the left predicate, and the output of the right predicate.

**type-synonym**  $'\alpha \text{ merge} = (('\alpha, '\alpha, '\alpha) \text{ mrg}, '\alpha) \text{ urel}$

*skip* is the merge predicate which ignores the output of both parallel predicates

**definition**  $\text{skip}_m :: '\alpha \text{ merge}$  **where**  
[*upred-defs*]:  $\text{skip}_m = (\$ \mathbf{v}' =_u \$ \mathbf{v}_<)$

*swap* is a predicate that the swaps the left and right indices; it is used to specify commutativity of the parallel operator

**definition**  $\text{swap}_m :: (('\alpha, '\beta, '\beta) \text{ mrg}) \text{ hrel}$  **where**  
[*upred-defs*]:  $\text{swap}_m = (0 - \mathbf{v}, 1 - \mathbf{v}) := (\& 1 - \mathbf{v}, \& 0 - \mathbf{v})$

A symmetric merge is one for which swapping the order of the merged concurrent predicates has no effect. We represent this by the following healthiness condition that states that  $\text{swap}_m$  is a left-unit.

**abbreviation**  $\text{SymMerge} :: '\alpha \text{ merge} \Rightarrow '\alpha \text{ merge}$  **where**  
 $\text{SymMerge}(M) \equiv (\text{swap}_m ;; M)$

## 27.3 Separating Simulations

$U0$  and  $U1$  are relations modify the variables of the input state-space such that they become indexed with 0 and 1, respectively.

**definition**  $U0 :: (' \beta_0, (' \alpha, '\beta_0, '\beta_1) \text{ mrg}) \text{ urel}$  **where**  
[*upred-defs*]:  $U0 = (\$ 0 - \mathbf{v}' =_u \$ \mathbf{v})$

**definition**  $U1 :: (' \beta_1, (' \alpha, '\beta_0, '\beta_1) \text{ mrg}) \text{ urel}$  **where**  
[*upred-defs*]:  $U1 = (\$ 1 - \mathbf{v}' =_u \$ \mathbf{v})$

**lemma**  $U0\text{-swap}: (U0 ;; \text{swap}_m) = U1$   
**by** (*rel-auto*)

**lemma**  $U1\text{-swap}: (U1 ;; \text{swap}_m) = U0$   
**by** (*rel-auto*)

As shown below, separating simulations can also be expressed using the following two alphabet extrusions

**definition**  $U0\alpha$  **where** [*upred-defs*]:  $U0\alpha = (1_L \times_L \text{mrg-left})$

**definition**  $U1\alpha$  **where** [*upred-defs*]:  $U1\alpha = (1_L \times_L \text{mrg-right})$

We then create the following intuitive syntax for separating simulations.

**abbreviation**  $U0\text{-alpha-lift } (\lceil - \rceil_0)$  **where**  $\lceil P \rceil_0 \equiv P \oplus_p U0\alpha$



**abbreviation**  $U1\text{-}\alpha\text{-lift}$  ( $\lceil \cdot \rceil_1$ ) **where**  $\lceil P \rceil_1 \equiv P \oplus_p U1\alpha$

$\lceil P \rceil_0$  is predicate  $P$  where all variables are indexed by 0, and  $\lceil P \rceil_1$  is where all variables are indexed by 1. We can thus equivalently express separating simulations using alphabet extrusion.

**lemma**  $U0\text{-}\alpha\text{-as-alpha}$ :  $(P ;; U0) = \lceil P \rceil_0$   
**by** (*rel-auto*)

**lemma**  $U1\text{-}\alpha\text{-as-alpha}$ :  $(P ;; U1) = \lceil P \rceil_1$   
**by** (*rel-auto*)

**lemma**  $U0\alpha\text{-vwb-lens}$  [*simp*]:  $vwb\text{-lens } U0\alpha$   
**by** (*simp add: U0 $\alpha$ -def id-vwb-lens prod-vwb-lens*)

**lemma**  $U1\alpha\text{-vwb-lens}$  [*simp*]:  $vwb\text{-lens } U1\alpha$   
**by** (*simp add: U1 $\alpha$ -def id-vwb-lens prod-vwb-lens*)

**lemma**  $U0\alpha\text{-indep-right-uvar}$  [*simp*]:  $vwb\text{-lens } x \implies U0\alpha \bowtie \text{out-var } (\text{right-uvar } x)$   
**by** (*force intro: plus-pres-lens-indep fst-snd-lens-indep lens-indep-left-comp*  
*simp add: U0 $\alpha$ -def right-uvar-def out-var-def prod-as-plus lens-comp-assoc[THEN sym]*)

**lemma**  $U1\alpha\text{-indep-left-uvar}$  [*simp*]:  $vwb\text{-lens } x \implies U1\alpha \bowtie \text{out-var } (\text{left-uvar } x)$   
**by** (*force intro: plus-pres-lens-indep fst-snd-lens-indep lens-indep-left-comp*  
*simp add: U1 $\alpha$ -def left-uvar-def out-var-def prod-as-plus lens-comp-assoc[THEN sym]*)

**lemma**  $U0\text{-}\alpha\text{-lift-bool-subst}$  [*usubst*]:  
 $\sigma(\$0 - x' \mapsto_s \text{true}) \dagger \lceil P \rceil_0 = \sigma \dagger \lceil P \llbracket \text{true} / \$x' \rrbracket \rceil_0$   
 $\sigma(\$0 - x' \mapsto_s \text{false}) \dagger \lceil P \rceil_0 = \sigma \dagger \lceil P \llbracket \text{false} / \$x' \rrbracket \rceil_0$   
**by** (*pred-auto+*)

**lemma**  $U1\text{-}\alpha\text{-lift-bool-subst}$  [*usubst*]:  
 $\sigma(\$1 - x' \mapsto_s \text{true}) \dagger \lceil P \rceil_1 = \sigma \dagger \lceil P \llbracket \text{true} / \$x' \rrbracket \rceil_1$   
 $\sigma(\$1 - x' \mapsto_s \text{false}) \dagger \lceil P \rceil_1 = \sigma \dagger \lceil P \llbracket \text{false} / \$x' \rrbracket \rceil_1$   
**by** (*pred-auto+*)

**lemma**  $U0\text{-}\alpha\text{-out-var}$  [*alpha*]:  $\lceil \$x' \rceil_0 = \$0 - x'$   
**by** (*rel-auto*)

**lemma**  $U1\text{-}\alpha\text{-out-var}$  [*alpha*]:  $\lceil \$x' \rceil_1 = \$1 - x'$   
**by** (*rel-auto*)

**lemma**  $U0\text{-skip}$  [*alpha*]:  $\lceil II \rceil_0 = (\$0 - \mathbf{v}' =_u \$\mathbf{v})$   
**by** (*rel-auto*)

**lemma**  $U1\text{-skip}$  [*alpha*]:  $\lceil II \rceil_1 = (\$1 - \mathbf{v}' =_u \$\mathbf{v})$   
**by** (*rel-auto*)

**lemma**  $U0\text{-segr}$  [*alpha*]:  $\lceil P ;; Q \rceil_0 = P ;; \lceil Q \rceil_0$   
**by** (*rel-auto*)

**lemma**  $U1\text{-segr}$  [*alpha*]:  $\lceil P ;; Q \rceil_1 = P ;; \lceil Q \rceil_1$   
**by** (*rel-auto*)

**lemma**  $U0\alpha\text{-comp-in-var}$  [*alpha*]:  $(\text{in-var } x) ;_L U0\alpha = \text{in-var } x$   
**by** (*simp add: U0 $\alpha$ -def alpha-in-var in-var-prod-lens pre-uvar-def*)

**lemma**  $U0\alpha\text{-comp-out-var } [\alpha]: (out\text{-var } x) ;_L U0\alpha = out\text{-var } (left\text{-uvar } x)$   
**by** (*simp add: U0 $\alpha$ -def alpha-out-var id-wb-lens left-uvar-def out-var-prod-lens*)

**lemma**  $U1\alpha\text{-comp-in-var } [\alpha]: (in\text{-var } x) ;_L U1\alpha = in\text{-var } x$   
**by** (*simp add: U1 $\alpha$ -def alpha-in-var in-var-prod-lens pre-uvar-def*)

**lemma**  $U1\alpha\text{-comp-out-var } [\alpha]: (out\text{-var } x) ;_L U1\alpha = out\text{-var } (right\text{-uvar } x)$   
**by** (*simp add: U1 $\alpha$ -def alpha-out-var id-wb-lens right-uvar-def out-var-prod-lens*)

## 27.4 Associative Merges

Associativity of a merge means that if we construct a three way merge from a two way merge and then rotate the three inputs of the merge to the left, then we get exactly the same three way merge back.

We first construct the operator that constructs the three way merge by effectively wiring up the two way merge in an appropriate way.

**definition**  $ThreeWayMerge :: 'a \text{ merge} \Rightarrow (('a, 'a, ('a, 'a, 'a) \text{ mrg}) \text{ mrg}, 'a) \text{ urel } (\mathbf{M}\beta'(-))$  **where**  
*[upred-defs]:*  $ThreeWayMerge \ M = ((\$0-\mathbf{v}' =_u \$0-\mathbf{v} \wedge \$1-\mathbf{v}' =_u \$1-0-\mathbf{v} \wedge \$\mathbf{v}_{<}' =_u \$\mathbf{v}_{<}) ;; M ;; U0 \wedge \$1-\mathbf{v}' =_u \$1-1-\mathbf{v} \wedge \$\mathbf{v}_{<}' =_u \$\mathbf{v}_{<}) ;; M$

The next definition rotates the inputs to a three way merge to the left one place.

**abbreviation**  $rotate_m$  **where**  $rotate_m \equiv (0-\mathbf{v}, 1-0-\mathbf{v}, 1-1-\mathbf{v}) := (\&1-0-\mathbf{v}, \&1-1-\mathbf{v}, \&0-\mathbf{v})$

Finally, a merge is associative if rotating the inputs does not effect the output.

**definition**  $AssocMerge :: 'a \text{ merge} \Rightarrow bool$  **where**  
*[upred-defs]:*  $AssocMerge \ M = (rotate_m ;; \mathbf{M}\beta(M) = \mathbf{M}\beta(M))$

## 27.5 Parallel Operators

We implement the following useful abbreviation for separating of two parallel processes and copying of the before variables, all to act as input to the merge predicate.

**abbreviation**  $par\text{-sep}$  (*infixr*  $\parallel_s$  85) **where**  
 $P \parallel_s Q \equiv (P ;; U0) \wedge (Q ;; U1) \wedge \$\mathbf{v}_{<}' =_u \$\mathbf{v}$

The following implementation of parallel by merge is less general than the book version, in that it does not properly partition the alphabet into two disjoint segments. We could actually achieve this specifying lenses into the larger alphabet, but this would complicate the definition of programs. May reconsider later.

**definition**  
 $par\text{-by-merge} :: ('a, 'b) \text{ urel} \Rightarrow (('a, 'b, 'c) \text{ mrg}, 'd) \text{ urel} \Rightarrow ('a, 'c) \text{ urel} \Rightarrow ('a, 'd) \text{ urel}$   
 $(- \parallel - \text{ [85,0,86] } 85)$   
**where** *[upred-defs]:*  $P \parallel_M Q = (P \parallel_s Q ;; M)$

**lemma**  $par\text{-by-merge-alt-def}: P \parallel_M Q = (\lceil P \rceil_0 \wedge \lceil Q \rceil_1 \wedge \$\mathbf{v}_{<}' =_u \$\mathbf{v}) ;; M$   
**by** (*simp add: par-by-merge-def U0-as-alpha U1-as-alpha*)

**lemma**  $shEx\text{-pbm-left}: ((\exists x \cdot P \ x) \parallel_M Q) = (\exists x \cdot (P \ x \parallel_M Q))$   
**by** (*rel-auto*)

**lemma**  $shEx\text{-pbm-right}: (P \parallel_M (\exists x \cdot Q \ x)) = (\exists x \cdot (P \parallel_M Q \ x))$   
**by** (*rel-auto*)

## 27.6 Unrestriction Laws

**lemma** *unrest-in-par-by-merge* [*unrest*]:  
 $\llbracket \$x \# P; \$x_{<} \# M; \$x \# Q \rrbracket \implies \$x \# P \parallel_M Q$   
**by** (*rel-auto*, *fastforce+*)

**lemma** *unrest-out-par-by-merge* [*unrest*]:  
 $\llbracket \$x' \# M \rrbracket \implies \$x' \# P \parallel_M Q$   
**by** (*rel-auto*)

## 27.7 Substitution laws

Substitution is a little tricky because when we push the expression through the composition operator the alphabet of the expression must also change. Consequently for now we only support literal substitution, though this could be generalised with suitable alphabet coercsions. We need quite a number of variants to support this which are below.

**lemma** *U0-seq-subst*:  $(P ;; U0) \llbracket \llbracket v \rrbracket / \$0 - x' \rrbracket = (P \llbracket \llbracket v \rrbracket / \$x' \rrbracket ;; U0)$   
**by** (*rel-auto*)

**lemma** *U1-seq-subst*:  $(P ;; U1) \llbracket \llbracket v \rrbracket / \$1 - x' \rrbracket = (P \llbracket \llbracket v \rrbracket / \$x' \rrbracket ;; U1)$   
**by** (*rel-auto*)

**lemma** *lit-pbm-subst* [*usubst*]:  
**fixes**  $x :: (- \implies 'a)$   
**shows**  
 $\bigwedge P Q M \sigma. \sigma(\$x \mapsto_s \llbracket v \rrbracket) \dagger (P \parallel_M Q) = \sigma \dagger ((P \llbracket \llbracket v \rrbracket / \$x \rrbracket) \parallel_{M \llbracket \llbracket v \rrbracket / \$x_{<} \rrbracket} (Q \llbracket \llbracket v \rrbracket / \$x \rrbracket))$   
 $\bigwedge P Q M \sigma. \sigma(\$x' \mapsto_s \llbracket v \rrbracket) \dagger (P \parallel_M Q) = \sigma \dagger (P \parallel_{M \llbracket \llbracket v \rrbracket / \$x' \rrbracket} Q)$   
**by** (*rel-auto*) $+$

**lemma** *bool-pbm-subst* [*usubst*]:  
**fixes**  $x :: (- \implies 'a)$   
**shows**  
 $\bigwedge P Q M \sigma. \sigma(\$x \mapsto_s \text{false}) \dagger (P \parallel_M Q) = \sigma \dagger ((P \llbracket \text{false} / \$x \rrbracket) \parallel_{M \llbracket \text{false} / \$x_{<} \rrbracket} (Q \llbracket \text{false} / \$x \rrbracket))$   
 $\bigwedge P Q M \sigma. \sigma(\$x \mapsto_s \text{true}) \dagger (P \parallel_M Q) = \sigma \dagger ((P \llbracket \text{true} / \$x \rrbracket) \parallel_{M \llbracket \text{true} / \$x_{<} \rrbracket} (Q \llbracket \text{true} / \$x \rrbracket))$   
 $\bigwedge P Q M \sigma. \sigma(\$x' \mapsto_s \text{false}) \dagger (P \parallel_M Q) = \sigma \dagger (P \parallel_{M \llbracket \text{false} / \$x' \rrbracket} Q)$   
 $\bigwedge P Q M \sigma. \sigma(\$x' \mapsto_s \text{true}) \dagger (P \parallel_M Q) = \sigma \dagger (P \parallel_{M \llbracket \text{true} / \$x' \rrbracket} Q)$   
**by** (*rel-auto*) $+$

**lemma** *zero-one-pbm-subst* [*usubst*]:  
**fixes**  $x :: (- \implies 'a)$   
**shows**  
 $\bigwedge P Q M \sigma. \sigma(\$x \mapsto_s 0) \dagger (P \parallel_M Q) = \sigma \dagger ((P \llbracket 0 / \$x \rrbracket) \parallel_{M \llbracket 0 / \$x_{<} \rrbracket} (Q \llbracket 0 / \$x \rrbracket))$   
 $\bigwedge P Q M \sigma. \sigma(\$x \mapsto_s 1) \dagger (P \parallel_M Q) = \sigma \dagger ((P \llbracket 1 / \$x \rrbracket) \parallel_{M \llbracket 1 / \$x_{<} \rrbracket} (Q \llbracket 1 / \$x \rrbracket))$   
 $\bigwedge P Q M \sigma. \sigma(\$x' \mapsto_s 0) \dagger (P \parallel_M Q) = \sigma \dagger (P \parallel_{M \llbracket 0 / \$x' \rrbracket} Q)$   
 $\bigwedge P Q M \sigma. \sigma(\$x' \mapsto_s 1) \dagger (P \parallel_M Q) = \sigma \dagger (P \parallel_{M \llbracket 1 / \$x' \rrbracket} Q)$   
**by** (*rel-auto*) $+$

**lemma** *numeral-pbm-subst* [*usubst*]:  
**fixes**  $x :: (- \implies 'a)$   
**shows**  
 $\bigwedge P Q M \sigma. \sigma(\$x \mapsto_s \text{numeral } n) \dagger (P \parallel_M Q) = \sigma \dagger ((P \llbracket \text{numeral } n / \$x \rrbracket) \parallel_{M \llbracket \text{numeral } n / \$x_{<} \rrbracket} (Q \llbracket \text{numeral } n / \$x \rrbracket))$   
 $\bigwedge P Q M \sigma. \sigma(\$x' \mapsto_s \text{numeral } n) \dagger (P \parallel_M Q) = \sigma \dagger (P \parallel_{M \llbracket \text{numeral } n / \$x' \rrbracket} Q)$

by (rel-auto)+

## 27.8 Parallel-by-merge laws

**lemma** *par-by-merge-false* [simp]:

$P \parallel_{\text{false}} Q = \text{false}$

by (rel-auto)

**lemma** *par-by-merge-left-false* [simp]:

$\text{false} \parallel_M Q = \text{false}$

by (rel-auto)

**lemma** *par-by-merge-right-false* [simp]:

$P \parallel_M \text{false} = \text{false}$

by (rel-auto)

**lemma** *par-by-merge-seq-add*:  $(P \parallel_M Q) ;; R = (P \parallel_M ;; R) Q$

by (simp add: par-by-merge-def seqr-assoc)

A skip parallel-by-merge yields a skip whenever the parallel predicates are both feasible.

**lemma** *par-by-merge-skip*:

assumes  $P ;; \text{true} = \text{true} Q ;; \text{true} = \text{true}$

shows  $P \parallel_{\text{skip}_m} Q = \text{II}$

using assms by (rel-auto)

**lemma** *skip-merge-swap*:  $\text{swap}_m ;; \text{skip}_m = \text{skip}_m$

by (rel-auto)

**lemma** *par-sep-swap*:  $P \parallel_s Q ;; \text{swap}_m = Q \parallel_s P$

by (rel-auto)

Parallel-by-merge commutes when the merge predicate is unchanged by swap

**lemma** *par-by-merge-commute-swap*:

shows  $P \parallel_M Q = Q \parallel_{\text{swap}_m} ;; M P$

**proof** –

have  $Q \parallel_{\text{swap}_m} ;; M P = (((Q ;; U0) \wedge (P ;; U1) \wedge \$\mathbf{v}_{<}{}' =_u \$\mathbf{v}) ;; \text{swap}_m) ;; M)$

by (simp add: par-by-merge-def seqr-assoc)

also have  $\dots = (((Q ;; U0 ;; \text{swap}_m) \wedge (P ;; U1 ;; \text{swap}_m) \wedge \$\mathbf{v}_{<}{}' =_u \$\mathbf{v}) ;; M)$

by (rel-auto)

also have  $\dots = (((Q ;; U1) \wedge (P ;; U0) \wedge \$\mathbf{v}_{<}{}' =_u \$\mathbf{v}) ;; M)$

by (simp add: U0-swap U1-swap)

also have  $\dots = P \parallel_M Q$

by (simp add: par-by-merge-def utp-pred-laws.inf.left-commute)

finally show ?thesis ..

qed

**theorem** *par-by-merge-commute*:

assumes  $M$  is *SymMerge*

shows  $P \parallel_M Q = Q \parallel_M P$

by (metis Healthy-if assms par-by-merge-commute-swap)

**lemma** *par-by-merge-mono-1*:

assumes  $P_1 \sqsubseteq P_2$

shows  $P_1 \parallel_M Q \sqsubseteq P_2 \parallel_M Q$

using assms by (rel-auto)

**lemma** *par-by-merge-mono-2*:

**assumes**  $Q_1 \sqsubseteq Q_2$   
**shows**  $(P \parallel_M Q_1) \sqsubseteq (P \parallel_M Q_2)$   
**using** *assms* **by** (*rel-blast*)

**lemma** *par-by-merge-mono*:

**assumes**  $P_1 \sqsubseteq P_2$   $Q_1 \sqsubseteq Q_2$   
**shows**  $P_1 \parallel_M Q_1 \sqsubseteq P_2 \parallel_M Q_2$   
**by** (*meson* *assms* *dual-order.trans* *par-by-merge-mono-1* *par-by-merge-mono-2*)

**theorem** *par-by-merge-assoc*:

**assumes** *M* is *SymMerge* *AssocMerge* *M*  
**shows**  $(P \parallel_M Q) \parallel_M R = P \parallel_M (Q \parallel_M R)$

**proof** –

**have**  $(P \parallel_M Q) \parallel_M R = ((P ;; U0) \wedge (Q ;; U0 ;; U1) \wedge (R ;; U1 ;; U1) \wedge \$\mathbf{v}_{<}' =_u \$\mathbf{v}) ;; \mathbf{M}\beta(M)$   
**by** (*rel-blast*)  
**also have**  $\dots = ((P ;; U0) \wedge (Q ;; U0 ;; U1) \wedge (R ;; U1 ;; U1) \wedge \$\mathbf{v}_{<}' =_u \$\mathbf{v}) ;; \text{rotate}_m ;; \mathbf{M}\beta(M)$   
**using** *AssocMerge-def* *assms*(2) **by** *force*  
**also have**  $\dots = ((Q ;; U0) \wedge (R ;; U0 ;; U1) \wedge (P ;; U1 ;; U1) \wedge \$\mathbf{v}_{<}' =_u \$\mathbf{v}) ;; \mathbf{M}\beta(M)$   
**by** (*rel-blast*)  
**also have**  $\dots = (Q \parallel_M R) \parallel_M P$   
**by** (*rel-blast*)  
**also have**  $\dots = P \parallel_M (Q \parallel_M R)$   
**by** (*simp* *add: assms*(1) *par-by-merge-commute*)  
**finally show** *?thesis* .

**qed**

**theorem** *par-by-merge-choice-left*:

$(P \sqcap Q) \parallel_M R = (P \parallel_M R) \sqcap (Q \parallel_M R)$   
**by** (*rel-auto*)

**theorem** *par-by-merge-choice-right*:

$P \parallel_M (Q \sqcap R) = (P \parallel_M Q) \sqcap (P \parallel_M R)$   
**by** (*rel-auto*)

**theorem** *par-by-merge-or-left*:

$(P \vee Q) \parallel_M R = (P \parallel_M R) \vee (Q \parallel_M R)$   
**by** (*rel-auto*)

**theorem** *par-by-merge-or-right*:

$P \parallel_M (Q \vee R) = (P \parallel_M Q) \vee (P \parallel_M R)$   
**by** (*rel-auto*)

**theorem** *par-by-merge-USUP-mem-left*:

$(\bigcap i \in I \cdot P(i)) \parallel_M Q = (\bigcap i \in I \cdot P(i) \parallel_M Q)$   
**by** (*rel-auto*)

**theorem** *par-by-merge-USUP-ind-left*:

$(\bigcap i \cdot P(i)) \parallel_M Q = (\bigcap i \cdot P(i) \parallel_M Q)$   
**by** (*rel-auto*)

**theorem** *par-by-merge-USUP-mem-right*:

$P \parallel_M (\bigcap i \in I \cdot Q(i)) = (\bigcap i \in I \cdot P \parallel_M Q(i))$   
**by** (*rel-auto*)

**theorem** *par-by-merge-USUP-ind-right*:  
 $P \parallel_M (\prod i \cdot Q(i)) = (\prod i \cdot P \parallel_M Q(i))$   
**by** (*rel-auto*)

## 27.9 Example: Simple State-Space Division

The following merge predicate divides the state space using a pair of independent lenses.

**definition** *StateMerge* ::  $('a \implies 'α) \Rightarrow ('b \implies 'α) \Rightarrow 'α \text{ merge } (M[-]_{\sigma})$  **where**  
 $[upred-defs]: M[a|b]_{\sigma} = (\$v' =_u (\$v_{<} \oplus \$0 - v \text{ on } \&a) \oplus \$1 - v \text{ on } \&b)$

**lemma** *swap-StateMerge*:  $a \bowtie b \implies (swap_m ;; M[a|b]_{\sigma}) = M[b|a]_{\sigma}$   
**by** (*rel-auto*, *simp-all add: lens-indep-comm*)

**abbreviation** *StateParallel* ::  $'α \text{ hrel} \Rightarrow ('a \implies 'α) \Rightarrow ('b \implies 'α) \Rightarrow 'α \text{ hrel} \Rightarrow 'α \text{ hrel } (-|-)_{\sigma}$  -  
 $[85,0,0,86] \ 86)$   
**where**  $P \mid a|b|_{\sigma} Q \equiv P \parallel_M [a|b]_{\sigma} Q$

**lemma** *StateParallel-commute*:  $a \bowtie b \implies P \mid a|b|_{\sigma} Q = Q \mid b|a|_{\sigma} P$   
**by** (*metis par-by-merge-commute-swap swap-StateMerge*)

**lemma** *StateParallel-form*:  
 $P \mid a|b|_{\sigma} Q = (\exists (st_0, st_1) \cdot P[\ll st_0 \gg / \$v'] \wedge Q[\ll st_1 \gg / \$v'] \wedge \$v' =_u (\$v \oplus \ll st_0 \gg \text{ on } \&a) \oplus \ll st_1 \gg \text{ on } \&b)$   
**by** (*rel-auto*)

**lemma** *StateParallel-form'*:  
**assumes** *vwb-lens a vwb-lens b a*  $\bowtie$  *b*  
**shows**  $P \mid a|b|_{\sigma} Q = \{\&a, \&b\} : [P \upharpoonright_v \{\$v, \$a'\} \wedge (Q \upharpoonright_v \{\$v, \$b'\})]$   
**using** *assms*  
**apply** (*simp add: StateParallel-form, rel-auto*)  
**apply** (*metis vwb-lens-wb wb-lens-axioms-def wb-lens-def*)  
**apply** (*metis vwb-lens-wb wb-lens.get-put*)  
**apply** (*simp add: lens-indep-comm*)  
**apply** (*metis (no-types, hide-lams) lens-indep-comm vwb-lens-wb wb-lens-def weak-lens.put-get*)  
**done**

We can frame all the variables that the parallel operator refers to

**lemma** *StateParallel-frame*:  
**assumes** *vwb-lens a vwb-lens b a*  $\bowtie$  *b*  
**shows**  $\{\&a, \&b\} : [P \mid a|b|_{\sigma} Q] = P \mid a|b|_{\sigma} Q$   
**using** *assms*  
**apply** (*simp add: StateParallel-form, rel-auto*)  
**using** *lens-indep-comm* **apply** *fastforce+*  
**done**

Parallel Hoare logic rule. This employs something similar to separating conjunction in the postcondition, but we explicitly require that the two conjuncts only refer to variables on the left and right of the parallel composition explicitly.

**theorem** *StateParallel-hoare* [*hoare*]:  
**assumes**  $\{c\} P \{d_1\}_u \{c\} Q \{d_2\}_u \ a \bowtie b \ a \dashv d_1 \ b \dashv d_2$   
**shows**  $\{c\} P \mid a|b|_{\sigma} Q \{d_1 \wedge d_2\}_u$   
**proof** –  
— Parallelise the specification

```

from assms(4,5)
have 1: ( $\lceil c \rceil_{<} \Rightarrow \lceil d_1 \wedge d_2 \rceil_{>}$ )  $\sqsubseteq$  ( $\lceil c \rceil_{<} \Rightarrow \lceil d_1 \rceil_{>}$ )  $|a|b|_{\sigma}$  ( $\lceil c \rceil_{<} \Rightarrow \lceil d_2 \rceil_{>}$ ) (is ?lhs  $\sqsubseteq$  ?rhs)
  by (simp add: StateParallel-form, rel-auto,metis assms(3) lens-indep-comm)
— Prove Hoare rule by monotonicity of parallelism
have 2:?rhs  $\sqsubseteq$   $P \mid a \mid b \mid_{\sigma} Q$ 
proof (rule par-by-merge-mono)
  show ( $\lceil c \rceil_{<} \Rightarrow \lceil d_1 \rceil_{>}$ )  $\sqsubseteq$   $P$ 
    using assms(1) hoare-r-def by auto
  show ( $\lceil c \rceil_{<} \Rightarrow \lceil d_2 \rceil_{>}$ )  $\sqsubseteq$   $Q$ 
    using assms(2) hoare-r-def by auto
qed
show ?thesis
  unfolding hoare-r-def using 1 2 order-trans by auto
qed

```

Specialised version of the above law where an invariant expression referring to variables outside the frame is preserved.

```

theorem StateParallel-frame-hoare [hoare]:
  assumes vwb-lens a vwb-lens b  $a \bowtie b$   $a \nabla d_1$   $b \nabla d_2$   $a \# c_1$   $b \# c_1$   $\{c_1 \wedge c_2\} P \{d_1\}_u \{c_1 \wedge c_2\} Q \{d_2\}_u$ 
  shows  $\{c_1 \wedge c_2\} P \mid a \mid b \mid_{\sigma} Q \{c_1 \wedge d_1 \wedge d_2\}_u$ 
proof —
  have  $\{c_1 \wedge c_2\} \{\&a, \&b\} : [P \mid a \mid b \mid_{\sigma} Q] \{c_1 \wedge d_1 \wedge d_2\}_u$ 
    by (auto intro!: frame-hoare-r' StateParallel-hoare simp add: assms unrest plus-vwb-lens)
  thus ?thesis
    by (simp add: StateParallel-frame assms)
qed
end

```

## 28 Meta-theory for the Standard Core

```

theory utp
imports
  utp-var
  utp-expr
  utp-expr-insts
  utp-expr-funcs
  utp-unrest
  utp-usedby
  utp-subst
  utp-meta-subst
  utp-alphabet
  utp-lift
  utp-pred
  utp-pred-laws
  utp-recursion
  utp-dynlog
  utp-rel
  utp-rel-laws
  utp-sequent
  utp-state-parser
  utp-sym-eval
  utp-tactics
  utp-hoare
  utp-wp

```

```

    utp-sp
    utp-theory
    utp-concurrency
    utp-rel-opsem
begin end

```

## 29 Overloaded Expression Constructs

```

theory utp-expr-ovld
  imports utp
begin

```

### 29.1 Overloadable Constants

For convenience, we often want to utilise the same expression syntax for multiple constructs. This can be achieved using ad-hoc overloading. We create a number of polymorphic constants and then overload their definitions using appropriate implementations. In order for this to work, each collection must have its own unique type. Thus we do not use the HOL map type directly, but rather our own partial function type, for example.

```

consts
  — Empty elements, for example empty set, nil list, 0...
  uempty    :: 'f
  — Function application, map application, list application...
  uapply    :: 'f ⇒ 'k ⇒ 'v
  — Function update, map update, list update...
  wupd      :: 'f ⇒ 'k ⇒ 'v ⇒ 'f
  — Domain of maps, lists...
  udom      :: 'f ⇒ 'a set
  — Range of maps, lists...
  uran      :: 'f ⇒ 'b set
  — Domain restriction
  udomres   :: 'a set ⇒ 'f ⇒ 'f
  — Range restriction
  uranres   :: 'f ⇒ 'b set ⇒ 'f
  — Collection cardinality
  ucard     :: 'f ⇒ nat
  — Collection summation
  usums     :: 'f ⇒ 'a
  — Construct a collection from a list of entries
  uentries  :: 'k set ⇒ ('k ⇒ 'v) ⇒ 'f

```

We need a function corresponding to function application in order to overload.

```

definition fun-apply :: ('a ⇒ 'b) ⇒ ('a ⇒ 'b)
where fun-apply f x = f x

```

```

declare fun-apply-def [simp]

```

```

definition ffun-entries :: 'k set ⇒ ('k ⇒ 'v) ⇒ ('k, 'v) ffun where
ffun-entries d f = graph-ffun {(k, f k) | k. k ∈ d}

```

We then set up the overloading for a number of useful constructs for various collections.

```

adhoc-overloading
  uempty 0 and

```



*uapply fun-apply and uapply nth and uapply pfun-app and*  
*uapply ffun-app and*  
*uupd pfun-upd and uupd ffun-upd and uupd list-augment and*  
*udom Domain and udom pdom and udom fdom and udom seq-dom and*  
*udom Range and uran pran and uran fran and uran set and*  
*udomres pdom-res and udomres fdom-res and*  
*uranres pran-res and udomres fran-res and*  
*ucard card and ucard pcard and ucard length and*  
*usums list-sum and usums Sum and usums pfun-sum and*  
*uentries pfun-entries and uentries ffun-entries*

## 29.2 Syntax Translations

### syntax

*-uundef*  $:: \text{logic } (\perp_u)$   
*-umap-empty*  $:: \text{logic } ([\ ]_u)$   
*-uapply*  $:: ('a \Rightarrow 'b, 'a) \text{ uexpr} \Rightarrow \text{utuple-args} \Rightarrow ('b, 'a) \text{ uexpr } (-'(-)_a [999,0] 999)$   
*-umaplet*  $:: [\text{logic}, \text{logic}] \Rightarrow \text{umaplet } (- / \mapsto / -)$   
 $\quad \quad \quad :: \text{umaplet} \Rightarrow \text{umaplets} \quad (-)$   
*-UMaplets*  $:: [\text{umaplet}, \text{umaplets}] \Rightarrow \text{umaplets } (- / -)$   
*-UMapUpd*  $:: [\text{logic}, \text{umaplets}] \Rightarrow \text{logic } (-'(-)_u [900,0] 900)$   
*-UMap*  $:: \text{umaplets} \Rightarrow \text{logic } ((1[\ ]_u))$   
*-ucard*  $:: \text{logic} \Rightarrow \text{logic } (\#_u'(-))$   
*-udom*  $:: \text{logic} \Rightarrow \text{logic } (\text{dom}_u'(-))$   
*-uran*  $:: \text{logic} \Rightarrow \text{logic } (\text{ran}_u'(-))$   
*-usum*  $:: \text{logic} \Rightarrow \text{logic } (\text{sum}_u'(-))$   
*-udom-res*  $:: \text{logic} \Rightarrow \text{logic} \Rightarrow \text{logic } (\text{infixl } \triangleleft_u 85)$   
*-uran-res*  $:: \text{logic} \Rightarrow \text{logic} \Rightarrow \text{logic } (\text{infixl } \triangleright_u 85)$   
*-uentries*  $:: \text{logic} \Rightarrow \text{logic} \Rightarrow \text{logic } (\text{entr}_u'(-, -))$

### translations

— Pretty printing for adhoc-overloaded constructs

$f(x)_a \leq \text{CONST } uapply\ f\ x$   
 $\text{dom}_u(f) \leq \text{CONST } udom\ f$   
 $\text{ran}_u(f) \leq \text{CONST } uran\ f$   
 $A \triangleleft_u f \leq \text{CONST } udomres\ A\ f$   
 $f \triangleright_u A \leq \text{CONST } uranres\ f\ A$   
 $\#_u(f) \leq \text{CONST } ucard\ f$   
 $f(k \mapsto v)_u \leq \text{CONST } uupd\ f\ k\ v$   
 $0 \leq \text{CONST } uempty$  — We have to do this so we don't see uempty. Is there a better way of printing?

— Overloaded construct translations

$f(x,y,z,u)_a == \text{CONST } bop\ \text{CONST } uapply\ f\ (x,y,z,u)_u$   
 $f(x,y,z)_a == \text{CONST } bop\ \text{CONST } uapply\ f\ (x,y,z)_u$   
 $f(x,y)_a == \text{CONST } bop\ \text{CONST } uapply\ f\ (x,y)_u$   
 $f(x)_a == \text{CONST } bop\ \text{CONST } uapply\ f\ x$   
 $\#_u(xs) == \text{CONST } uop\ \text{CONST } ucard\ xs$   
 $\text{sum}_u(A) == \text{CONST } uop\ \text{CONST } usums\ A$   
 $\text{dom}_u(f) == \text{CONST } uop\ \text{CONST } udom\ f$   
 $\text{ran}_u(f) == \text{CONST } uop\ \text{CONST } uran\ f$   
 $[\ ]_u \Rightarrow \ll \text{CONST } uempty \gg$   
 $\perp_u == \ll \text{CONST } undefined \gg$   
 $A \triangleleft_u f == \text{CONST } bop\ (\text{CONST } udomres)\ A\ f$   
 $f \triangleright_u A == \text{CONST } bop\ (\text{CONST } uranres)\ f\ A$   
 $\text{entr}_u(d,f) == \text{CONST } bop\ \text{CONST } uentries\ d\ \ll f \gg$   
 $-UMapUpd\ m\ (-UMaplets\ xy\ ms) == -UMapUpd\ (-UMapUpd\ m\ xy)\ ms$

$-UMapUpd\ m\ (-umaplet\ x\ y) == CONST\ trop\ CONST\ uupd\ m\ x\ y$   
 $-UMap\ ms == -UMapUpd\ []_u\ ms$   
 $-UMap\ (-UMaplets\ ms1\ ms2) <= -UMapUpd\ (-UMap\ ms1)\ ms2$   
 $-UMaplets\ ms1\ (-UMaplets\ ms2\ ms3) <= -UMaplets\ (-UMaplets\ ms1\ ms2)\ ms3$

### 29.3 Simplifications

**lemma** *ufun-apply-lit* [*simp*]:

$\llbracket f \rrbracket (\llbracket x \rrbracket)_a = \llbracket f(x) \rrbracket$   
**by** (*transfer*, *simp*)

**lemma** *lit-plus-appl* [*lit-norm*]:  $\llbracket (+) \rrbracket (x)_a (y)_a = x + y$  **by** (*simp add: uepr-defs, transfer, simp*)

**lemma** *lit-minus-appl* [*lit-norm*]:  $\llbracket (-) \rrbracket (x)_a (y)_a = x - y$  **by** (*simp add: uepr-defs, transfer, simp*)

**lemma** *lit-mult-appl* [*lit-norm*]:  $\llbracket \text{times} \rrbracket (x)_a (y)_a = x * y$  **by** (*simp add: uepr-defs, transfer, simp*)

**lemma** *lit-divide-apply* [*lit-norm*]:  $\llbracket (/) \rrbracket (x)_a (y)_a = x / y$  **by** (*simp add: uepr-defs, transfer, simp*)

**lemma** *pfun-entries-apply* [*simp*]:

$(entr_u(d, f) :: ((k, 'v)\ pfun, 'α)\ uepr)(i)_a = ((\llbracket f \rrbracket (i)_a) \triangleleft i \in_u d \triangleright \perp_u)$   
**by** (*pred-auto*)

**lemma** *udom-uupdate-pfun* [*simp*]:

**fixes**  $m :: ((k, 'v)\ pfun, 'α)\ uepr$   
**shows**  $dom_u(m(k \mapsto v)_u) = \{k\}_u \cup_u dom_u(m)$   
**by** (*rel-auto*)

**lemma** *uapply-uupdate-pfun* [*simp*]:

**fixes**  $m :: ((k, 'v)\ pfun, 'α)\ uepr$   
**shows**  $(m(k \mapsto v)_u)(i)_a = v \triangleleft i =_u k \triangleright m(i)_a$   
**by** (*rel-auto*)

### 29.4 Indexed Assignment

**syntax**

— Indexed assignment  
 $-assignment-upd :: svid \Rightarrow uepr \Rightarrow uepr \Rightarrow logic\ (([-] := / -) [63, 0, 0] 62)$

**translations**

— Indexed assignment uses the overloaded collection update function *uupd*.  
 $-assignment-upd\ x\ k\ v \Rightarrow x := \&x(k \mapsto v)_u$

**end**

## 30 Meta-theory for the Standard Core with Overloaded Constructs

**theory** *utp-full*

**imports** *utp utp-expr-ovld*

**begin end**

## 31 UTP Easy Expression Parser

**theory** *utp-easy-parser*

**imports** *utp-full*

**begin**

### 31.1 Replacing the Expression Grammar

The following theory provides an easy to use expression parser that is primarily targetted towards expressing programs. Unlike the built-in UTP expression syntax, this uses a closed grammar separate to the HOL *logic* nonterminal, that gives more freedom in what can be expressed. In particular, identifiers are interpreted as UTP variables rather than HOL variables and functions do not require subscripts and other strange decorations.

The first step is to remove the from the UTP parse the following grammar rule that uses arbitrary HOL logic to represent expressions. Instead, we will populate the *uexp* grammar manually.

#### purge-syntax

*-uexp-l* :: *logic*  $\Rightarrow$  *uexp* (- [64] 64)

### 31.2 Expression Operators

#### syntax

*-ue-quote* :: *uexp*  $\Rightarrow$  *logic* ('(-')<sub>e</sub>)  
*-ue-tuple* :: *uexprs*  $\Rightarrow$  *uexp* ('(-')<sub>t</sub>)  
*-ue-lit* :: *logic*  $\Rightarrow$  *uexp* ( $\llcorner$ - $\gg$ )  
*-ue-var* :: *svid*  $\Rightarrow$  *uexp* (-)  
*-ue-eq* :: *uexp*  $\Rightarrow$  *uexp*  $\Rightarrow$  *uexp* (**infix** = 150)  
*-ue-uop* :: *id*  $\Rightarrow$  *uexp*  $\Rightarrow$  *uexp* (-'(-') [999,0] 999)  
*-ue-bop* :: *id*  $\Rightarrow$  *uexp*  $\Rightarrow$  *uexp*  $\Rightarrow$  *uexp* (-'(-, -') [999,0,0] 999)  
*-ue-trop* :: *id*  $\Rightarrow$  *uexp*  $\Rightarrow$  *uexp*  $\Rightarrow$  *uexp*  $\Rightarrow$  *uexp* (-'(-, -, -') [999,0,0,0] 999)  
*-ue-apply* :: *uexp*  $\Rightarrow$  *uexp*  $\Rightarrow$  *uexp* (-[-] [999] 999)

#### translations

*-ue-quote* *e*  $\Rightarrow$  *e*  
*-ue-tuple* (-*uexprs* *x* (-*uexprs* *y* *z*))  $\Rightarrow$  *-ue-tuple* (-*uexprs* *x* (-*ue-tuple* (-*uexprs* *y* *z*)))  
*-ue-tuple* (-*uexprs* *x* *y*)  $\Rightarrow$  *CONST* *bop* *CONST* *Pair* *x* *y*  
*-ue-tuple* *x*  $\Rightarrow$  *x*  
*-ue-lit* *x*  $\Rightarrow$  *CONST* *lit* *x*  
*-ue-var* *x*  $\Rightarrow$  *CONST* *utp-expr.var* (*CONST* *pr-var* *x*)  
*-ue-eq* *x* *y*  $\Rightarrow$  *x* =<sub>u</sub> *y*  
*-ue-uop* *f* *x*  $\Rightarrow$  *CONST* *uop* *f* *x*  
*-ue-bop* *f* *x* *y*  $\Rightarrow$  *CONST* *bop* *f* *x* *y*  
*-ue-trop* *f* *x* *y*  $\Rightarrow$  *CONST* *trop* *f* *x* *y*  
*-ue-apply* *f* *x*  $\Rightarrow$  *f*(*x*)<sub>a</sub>

### 31.3 Predicate Operators

#### syntax

*-ue-true* :: *uexp* (*true*)  
*-ue-false* :: *uexp* (*false*)  
*-ue-not* :: *uexp*  $\Rightarrow$  *uexp* ( $\neg$  - [40] 40)  
*-ue-conj* :: *uexp*  $\Rightarrow$  *uexp*  $\Rightarrow$  *uexp* (**infixr**  $\wedge$  135)  
*-ue-disj* :: *uexp*  $\Rightarrow$  *uexp*  $\Rightarrow$  *uexp* (**infixr**  $\vee$  130)  
*-ue-impl* :: *uexp*  $\Rightarrow$  *uexp*  $\Rightarrow$  *uexp* (**infixr**  $\Rightarrow$  125)  
*-ue-iff* :: *uexp*  $\Rightarrow$  *uexp*  $\Rightarrow$  *uexp* (**infixr**  $\Leftrightarrow$  125)  
*-ue-mem* :: *uexp*  $\Rightarrow$  *uexp*  $\Rightarrow$  *uexp* ((-/  $\in$  -) [151, 151] 150)  
*-ue-nmem* :: *uexp*  $\Rightarrow$  *uexp*  $\Rightarrow$  *uexp* ((-/  $\notin$  -) [151, 151] 150)

#### translations

*-ue-true*  $\Rightarrow$  *CONST* *true-upred*

$-ue\text{-false} \Rightarrow \text{CONST false-upred}$   
 $-ue\text{-not } p \Rightarrow \text{CONST not-upred } p$   
 $-ue\text{-conj } p \ q \Rightarrow p \wedge_p q$   
 $-ue\text{-disj } p \ q \Rightarrow p \vee_p q$   
 $-ue\text{-impl } p \ q \Rightarrow p \Rightarrow q$   
 $-ue\text{-iff } p \ q \Rightarrow p \Leftrightarrow q$   
 $-ue\text{-mem } x \ A \Rightarrow x \in_u A$   
 $-ue\text{-nmem } x \ A \Rightarrow x \notin_u A$

### 31.4 Arithmetic Operators

#### syntax

$-ue\text{-num} \quad :: \text{num-const} \Rightarrow \text{uexp } (-)$   
 $-ue\text{-size} \quad :: \text{uexp} \Rightarrow \text{uexp } (\#- [999] \ 999)$   
 $-ue\text{-eq} \quad :: \text{uexp} \Rightarrow \text{uexp} \Rightarrow \text{uexp } (\text{infix} = 150)$   
 $-ue\text{-le} \quad :: \text{uexp} \Rightarrow \text{uexp} \Rightarrow \text{uexp } (\text{infix} \leq 150)$   
 $-ue\text{-lt} \quad :: \text{uexp} \Rightarrow \text{uexp} \Rightarrow \text{uexp } (\text{infix} < 150)$   
 $-ue\text{-ge} \quad :: \text{uexp} \Rightarrow \text{uexp} \Rightarrow \text{uexp } (\text{infix} \geq 150)$   
 $-ue\text{-gt} \quad :: \text{uexp} \Rightarrow \text{uexp} \Rightarrow \text{uexp } (\text{infix} > 150)$   
 $-ue\text{-zero} \quad :: \text{uexp } (0)$   
 $-ue\text{-one} \quad :: \text{uexp } (1)$   
 $-ue\text{-plus} \quad :: \text{uexp} \Rightarrow \text{uexp} \Rightarrow \text{uexp } (\text{infixl} + 165)$   
 $-ue\text{-uminus} :: \text{uexp} \Rightarrow \text{uexp } (- - [181] \ 180)$   
 $-ue\text{-minus} \quad :: \text{uexp} \Rightarrow \text{uexp} \Rightarrow \text{uexp } (\text{infixl} - 165)$   
 $-ue\text{-times} \quad :: \text{uexp} \Rightarrow \text{uexp} \Rightarrow \text{uexp } (\text{infixl} * 170)$   
 $-ue\text{-div} \quad :: \text{uexp} \Rightarrow \text{uexp} \Rightarrow \text{uexp } (\text{infixl} \text{ div } 170)$

#### translations

$-ue\text{-num } x \Rightarrow \text{-Numeral } x$   
 $-ue\text{-size } e \Rightarrow \#_u(e)$   
 $-ue\text{-le } x \ y \Rightarrow x \leq_u y$   
 $-ue\text{-lt } x \ y \Rightarrow x <_u y$   
 $-ue\text{-ge } x \ y \Rightarrow x \geq_u y$   
 $-ue\text{-gt } x \ y \Rightarrow x >_u y$   
 $-ue\text{-zero} \Rightarrow 0$   
 $-ue\text{-one} \Rightarrow 1$   
 $-ue\text{-plus } x \ y \Rightarrow x + y$   
 $-ue\text{-uminus } x \Rightarrow -x$   
 $-ue\text{-minus } x \ y \Rightarrow x - y$   
 $-ue\text{-times } x \ y \Rightarrow x * y$   
 $-ue\text{-div } x \ y \Rightarrow \text{CONST divide } x \ y$

### 31.5 Sets

#### syntax

$-ue\text{-empset} \quad :: \text{uexp } (\{\})$   
 $-ue\text{-setprod} \quad :: \text{uexp} \Rightarrow \text{uexp} \Rightarrow \text{uexp } (\text{infixr } \times 80)$   
 $-ue\text{-atLeastAtMost} \quad :: \text{uexp} \Rightarrow \text{uexp} \Rightarrow \text{uexp } ((1\{-..\})$   
 $-ue\text{-atLeastLessThan} :: \text{uexp} \Rightarrow \text{uexp} \Rightarrow \text{uexp } ((1\{-..<-\}))$

#### translations

$-ue\text{-empset} \Rightarrow \{\}_u$   
 $-ue\text{-setprod } e \ f \Rightarrow \text{CONST bop } (\text{CONST Product-Type.Times}) \ e \ f$   
 $-ue\text{-atLeastAtMost } m \ n \Rightarrow \{m..n\}_u$   
 $-ue\text{-atLeastLessThan } m \ n \Rightarrow \{m..<n\}_u$

## 31.6 Imperative Program Syntax

**syntax**

```
-ue-if-then  :: uexp ⇒ logic ⇒ logic ⇒ logic (if - then - else - fi)
-ue-hoare   :: uexp ⇒ logic ⇒ uexp ⇒ logic ({-} / - / {-})
-ue-wp      :: logic ⇒ uexp ⇒ uexp (infix wp 60)
```

**translations**

```
-ue-if-then b P Q => P ◁ b ▷r Q
-ue-hoare b P c => {b}P{c}u
-ue-wp P b => P wp b
```

**end**

## 32 Example: Summing a List

**theory** *sum-list*

**imports** *../utp-easy-parser*

**begin**

This theory exemplifies the use of the Isabelle/UTP Hoare logic verification component. We first create a state space with the variables the program needs.

**alphabet** *st-sum-list* =

```
i  :: nat
xs :: int list
ans :: int
```

Next, we define the program as by a homogeneous relation over the state-space type.

**abbreviation** *Sum-List* :: *st-sum-list* *hrel* **where**

```
Sum-List ≡
i := 0 ;;
ans := 0 ;;
while (i < #xs) invr (ans = list-sum(take(i, xs)))
do
  ans := ans + xs[i] ;;
  i := i + 1
od
```

Next, we symbolically evaluate some examples.

**lemma** *TRY* ( $[\&xs \mapsto_s \ll[4,3,7,1,12,8]\gg] \models \text{Sum-List}$ )

**apply** (*sym-eval*) **oops**

Finally, we verify the program.

**theorem** *Sum-List-sums*:

```
{xs = <<XS>>}} Sum-List {{ans = list-sum(xs)}}
by (hoare-auto, metis add.foldr-snoc take-Suc-conv-app-nth)
```

**end**

## 33 Simple UTP real-time theory

**theory** *utp-simple-time* **imports** *../utp* **begin**

In this section we give a small example UTP theory, and show how Isabelle/UTP can be used to automate production of programming laws.

### 33.1 Observation Space and Signature

We first declare the observation space for our theory of timed relations. It consists of two variables, to denote time and the program state, respectively.

**alphabet** *'s st-time* =  
*clock* :: nat *st* :: 's

A timed relation is a homogeneous relation over the declared observation space.

**type-synonym** *'s time-rel* = *'s st-time hrel*

We introduce the following operator for adding an  $n$ -unit delay to a timed relation.

**definition** *Wait* :: nat  $\Rightarrow$  *'s time-rel* **where**  
*[upred-defs]: Wait*( $n$ ) = ( $\$clock' =_u \$clock + \ll n \gg \wedge \$st' =_u \$st$ )

### 33.2 UTP Theory

We define a single healthiness condition which ensures that the clock monotonically advances, and so forbids reverse time travel.

**definition** *HT* :: *'s time-rel*  $\Rightarrow$  *'s time-rel* **where**  
*[upred-defs]: HT*( $P$ ) = ( $P \wedge \$clock \leq_u \$clock'$ )

This healthiness condition is idempotent, monotonic, and also continuous, meaning it distributes through arbitrary non-empty infima.

**theorem** *HT-idem*: *HT*(*HT*( $P$ )) = *HT*( $P$ ) **by** *rel-auto*

**theorem** *HT-mono*:  $P \sqsubseteq Q \implies HT(P) \sqsubseteq HT(Q)$  **by** *rel-auto*

**theorem** *HT-continuous*: *Continuous HT* **by** *rel-auto*

We now create the UTP theory object for timed relations. This is done using a local interpretation *utp-theory-continuous HT*. This raises the proof obligations that *HT* is both idempotent and continuous, which we have proved already. The result of this command is a collection of theorems that can be derived from these facts. Notably, we obtain a complete lattice of timed relations via the Knaster-Tarski theorem. We also apply some locale rewrites so that the theorems that are exports have a more intuitive form.

**interpretation** *time-theory*: *utp-theory-continuous HT*  
**rewrites**  $P \in \text{carrier } \text{time-theory.thy-order} \longleftrightarrow P \text{ is } HT$   
**and**  $\text{carrier } \text{time-theory.thy-order} \rightarrow \text{carrier } \text{time-theory.thy-order} \equiv \llbracket HT \rrbracket_H \rightarrow \llbracket HT \rrbracket_H$   
**and**  $\text{le } \text{time-theory.thy-order} = (\sqsubseteq)$   
**and**  $\text{eq } \text{time-theory.thy-order} = (=)$   
**proof** –  
  **show** *utp-theory-continuous HT*  
  **proof**  
    **show**  $\bigwedge P. HT (HT P) = HT P$   
    **by** (*simp add: HT-idem*)  
    **show** *Continuous HT*  
    **by** (*simp add: HT-continuous*)  
  **qed**  
**qed** (*simp-all*)

The object *time-theory* is a new namespace that contains both definitions and theorems. Since the theory forms a complete lattice, we obtain a top element, bottom element, and a least fixed-point constructor. We give all of these some intuitive syntax.

**notation** *time-theory.utp-top* ( $\top_t$ )  
**notation** *time-theory.utp-bottom* ( $\perp_t$ )  
**notation** *time-theory.utp-lfp* ( $\mu_t$ )

Below is a selection of theorems that have been exported by the locale interpretation.

**thm** *time-theory.bottom-healthy*  
**thm** *time-theory.top-higher*  
**thm** *time-theory.meet-bottom*  
**thm** *time-theory.LFP-unfold*

### 33.3 Closure Laws

*HT* applied to *Wait* has no affect, since the latter always advances time.

**lemma** *HT-Wait*:  $HT(Wait(n)) = Wait(n)$  **by** (*rel-auto*)

**lemma** *HT-Wait-closed* [*closure*]: *Wait*(*n*) is *HT*  
**by** (*simp add: HT-Wait Healthy-def*)

Relational identity, *II*, is likewise *HT*-healthy.

**lemma** *HT-skip-closed* [*closure*]: *II* is *HT*  
**by** (*rel-auto*)

*HT* is closed under sequential composition, which can be shown by transitivity of ( $\leq$ ).

**lemma** *HT-seqr-closed* [*closure*]:  
 $\llbracket P \text{ is } HT; Q \text{ is } HT \rrbracket \implies P ;; Q \text{ is } HT$   
**by** (*rel-auto, meson dual-order.trans*) — Sledgehammer required

Assignment is also healthy, provided that the clock variable is not assigned.

**lemma** *HT-assign-closed* [*closure*]:  $\llbracket vwb\text{-}lens\ x; clock \bowtie x \rrbracket \implies x := v \text{ is } HT$   
**by** (*rel-auto, metis (mono-tags, lifting) eq-iff lens.select-convs(1) lens-indep-get st-time.select-convs(1)*)

An alternative characterisation of the above is that *x* is within the state space lens.

**lemma** *HT-assign-closed'* [*closure*]:  $\llbracket vwb\text{-}lens\ x; x \subseteq_L st \rrbracket \implies x := v \text{ is } HT$   
**by** (*rel-auto*)

### 33.4 Algebraic Laws

Finally, we prove some useful algebraic laws.

**theorem** *Wait-skip*:  $Wait(0) = II$  **by** (*rel-auto*)

**theorem** *Wait-Wait*:  $Wait(m) ;; Wait(n) = Wait(m + n)$  **by** (*rel-auto*)

**theorem** *Wait-cond*:  $Wait(m) ;; (P \triangleleft b \triangleright_r Q) = (Wait\ m ;; P) \triangleleft b \llbracket \&clock + \ll m \gg / \&clock \rrbracket \triangleright_r (Wait\ m ;; Q)$   
**by** (*rel-auto*)

**end**

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<sup>2</sup>CyPhyAssure Project: <https://www.cs.york.ac.uk/circus/CyPhyAssure/>

<sup>3</sup>RoboCalc Project: <https://www.cs.york.ac.uk/circus/RoboCalc/>



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